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Department of Mathematics

The Adomian Decomposition Method For Solving Partial Differential Equations

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Declaration

I certify that this thesis, submitted for the degree of Master in Mathematics to the Department of Mathematics at Birzeit University. And that this thesis (or any part of it) has not been submitted for a higher degree to any other university.

Eman Alawawdah　　　　Signature................

Abstract

We review the Adomian decomposition method(ADM). We give general description of this method, the convergence analysis of this method and we present some modifications for the standard ADM. In chapter 3, we will give examples of using ADM for obtaining exact and numerical solutions for nonlinear ordinary differential equations, partial differential equations and integral equations. We present a solution of generalized log-KdV (The Korteweg de Vries) equation as an application of using the ADM for solving nonlinear PDEs of higher order. In chapter 4, we will consider different types of inverse partial differential equations, boundary conditions identification, coefficient identification and source identification using ADM. In this part, we will try to solve the heat conduction inverse problem in special cases using Adomian decomposition method.

الملخص:

الهدف الرئيسي من هذه الرسالة هوعرض طريقة الفصلAdomian التي تستخدم في حل المعادلات التفاضلية العادية والجزئية وكذلك عرض بعض التحديثات التي طرأت على هذه الطريقة من أجل تسريع واختصار خطوات الحل.

في الفصل الثاني نقدم العديد من الامثلة على استخدام طريق الفصل Adomian في حل المعادلات التفاضلية العادية الخطة وغير الخطية والمعادلات التفاضلية الجزئية الخطية وغير الخطية وايضا أنظمة مكونة من هذه المعادلات. في الجزء الثاني نقوم بحل معادلة KDV من الدرجة الثالثة. الجزء الأخير من هذا الفصل يعرض كيفية حل المعادلات التفاضلية التكاملية.

الفصل الثالث والاخير نقوم بعرض أنواع عدة من المعادلات التفاضلية العكسية وحلها باستخدام طريقة الفصل Adomian ومن ثم نحاول حل مشكلة التوصيل العكسي بهذه الطريقة.

Contents

CHAPTER 1

Introduction

The Adomian decomposition method was presented in 1980's by Adomian. The method is very useful for solving linear and nonlinear ordinary and partial differential equations, algebraic equations, functional equations, integral differential equations and the convergence analysis of the ADM was discussed in [2]. Y. Cherruault and G. Adomian give the new proof of convergence analysis of the decomposition method [16]. E. Babolian And J. Biazar, define the order of the convergence of adomian method in [11]. After that many modifications were made on this method by numerous researchers in an attempt to improve the accuracy or extend the applications of this method. In given in [9]. A new modification methods of the ADM, Wazwaz modifications and the two step modified Adomian decomposition method. In chapter 3, we will use the ADM to solve different types of differential equations. Yahya Qaid Hasan and Liu Ming Zhu modified the ADM to solve second order singular initial value ordinary differential equations [21]. Several examples on solving the ordinary differential equations, initial value problems and boundary value problems are introduced in [2]. J. Biazar, E. Babolian and R. Islam in [12] obtained the solution of a system of ordinary differential equations by using ADM. In the second part of chapter, we will apply the ADM for solving partial differential equations. We will consider first order PDEs as done by [7]. Then, we move to several 2^{nd} order PDE's, linear heat equation, nonlinear heat equation [14], linear wave equation and nonlinear wave equation [15]. The generalized log-KdV(Diederik Korteweg and Gustav de Vries)equation in [38] will be solved by ADM as an application on higher order PDE's. We apply the method for solving system of PDE's as in [7]. At the end of this chapter, we show how we can solve the integral equations by using Adomian decomposition method.

In chapter 4, we review some inverse problems and show how ADM is used for solving these problems. There are many classifications of the inverse problems, we will deal with boundary conditions determination of inverse problems[31] and parameter determination for some equations[35].

CHAPTER 2

Adomian Decomposition Method(ADM)

2.1 General Description of ADM

In this section we give standard description of the ADM and some of its modifications depending on the references[7, 40, 8, 2]. Consider the general equation

$$Lu + Nu + Ru = g \tag{2.1.1}$$

where u is the unknown function, L is the linear differential operator of higher order which is easily invertible. Assume its inverse is L^{-1} and it will be an integral operator, N is the nonlinear operator, R is the remaining linear part and g is a given function (source). Take L^{-1} to both sides of (2.1.1) to get:

$$L^{-1}(Lu + Nu + Ru = g)$$

$$L^{-1}Lu = L^{-1}g - L^{-1}N(u) - L^{-1}R(u)$$

thus,

$$u - \phi = L^{-1}g - L^{-1}N(u) - L^{-1}R(u) \tag{2.1.2}$$

where ϕ is presented from the initial conditions or from the boundary conditions or both, it depends on how we choose differential operator that solve the given problem. The ADM assumes that solution u of the functional equation can be decomposed into infinite series

$$u = \sum_{n=0}^{\infty} u_n.$$

and the nonlinear term $N(u)$ can be written as infinite series $Nu = \sum_{n=0}^{\infty} A_n$ where the A_n's are the Adomian polynomials.By substitution this in (2.1.2) gives:

$$\sum_{n=0}^{\infty} u_n = \phi + L^{-1}g - L^{-1}\sum_{n=0}^{\infty} A_n - L^{-1}\sum_{n=0}^{\infty} R(u_n) \tag{2.1.3}$$

Now from equation (2.1.3), we can obtain the solution algorithm as follows:

$$u_0 = \phi + L^{-1}g, \quad u_{n+1} = -L^{-1}(A_n + Ru_n), \quad n = 0, 1, 2, \cdots \tag{2.1.4}$$

Given u_0, the other terms of u can be determined respectively. If one term of u_n is equal zero then the following terms are all zeros.

2.1.1 Solution algorithm

Refereing to [1, 17], we will show how the solution algorithm (2.1.4) was chosen. Consider the nonlinear functional equation

$$u - N(u) = f \tag{2.1.5}$$

where N is the nonlinear operator and f is a function determined after applying L^{-1} to the source function g. Suppose that the solution of (2.1.5) is a family of

$$u = \sum_{n=0}^{\infty} u_n \lambda^n \tag{2.1.6}$$

where λ is a parameter. Suppose that the radius of convergent ρ of the series above is grater than one, so the series converges for $|\lambda| < \rho$ where $\rho > 1$. As we showed previously the nonlinear function $N(u)$ can be expanded in infinite series

$$N(u) = \sum_{i=0}^{\infty} \alpha_i u^i \tag{2.1.7}$$

with radius of convergence $\rho_0 > 1$, this implies that the series above converges for $|u| < \rho_0$. In general it can suppose that $\rho_0 = \infty$ because in practical applications the nonlinear operator $N(u)$ is a polynomial or a nonlinear function admitting an entire series converging for any u with $|u| < \infty$. Now by substituting (2.1.6) in (2.1.7) we get

$$(a) \quad N(\sum_{n=0}^{\infty} u_n \lambda^n) = \sum_{i=0}^{\infty} \alpha_i (\sum_{n=0}^{\infty} u_n \lambda^n)^i$$

let $\sum_{i=0}^{\infty} \alpha_i (\sum_{n=0}^{\infty} u_n)^i = \sum_{i=0}^{\infty} A_i$, then the above series is equivalent to

$$(b) \quad N(u) = \sum_{i=0}^{\infty} A_i \lambda^i = A_0 + A_1\lambda + A_2\lambda^2 + A_3\lambda^3 + \cdots \tag{2.1.8}$$

From the first expansion (a) of $N(u_\lambda)$ in equation above we have
$N(u_\lambda) = \alpha_0 + \alpha_1(u_0 + u_1\lambda + u_2\lambda^2 + \cdots) + \alpha_2(u_0 + u_1\lambda + u_2\lambda^2 + \cdots)^2$
$+ \alpha_3(u_0 + u_1\lambda + u_2\lambda^2 + \cdots)^3 + \cdots$
$= \alpha_0 + \alpha_1 u_0 + \alpha_1 u_1 \lambda + \alpha_1 u_2 \lambda^2 + \cdots + \alpha_2 u_0^2 + 2\alpha_2 u_0 u_1 \lambda + \alpha_2 u_1^2 \lambda^2 + 2\alpha_2 u_0 u_2 \lambda^2 + \cdots +$
$\alpha_3 u_0^3 + 3\alpha_3 u_0^2 u_1 \lambda + \cdots$.
By matching this expansion with the second formula (b), the values of A_i's can be

obtained as follows

$$A_0 = \alpha_0 + \alpha_1 u_0 + \alpha_2 u_0^2 + \alpha_3 u_0^3 + \cdots + \alpha_n u_0^n + \cdots$$
$$A_1 = \alpha_1 u_1 + 2\alpha_2 u_0 u_1 + 2\alpha_3 u_0^2 u_1 + \cdots + n\alpha_n u_0^{n-1} u_1 + \cdots$$
$$A_2 = \alpha_1 u_2 + \alpha_2 u_1^2 + 2\alpha_2 u_0 u_1 + \alpha_3 u_0^2 u_2 + 3\alpha_3 u_0 u_1^2 + \cdots$$

From above we can conclude that the values of A_i's depend only on the values of u_i's.

By substituting (2.1.8) and (2.1.6) in (2.1.5) and make $\lambda = 1$, we obtain, because of the convergence of the two series:

$$\sum_{n=0}^{\infty} u_n - \sum_{i=0}^{\infty} A_i = f \tag{2.1.9}$$

This equation can be satisfied if:

$$u_0 = f$$

$$u_{n+1} = A_n(u_0, u_1, \ldots, u_n)$$

2.1.2 The Adomian polynomials

The Adomian polynomials A_n's are first constructed by Adomian in 1992, he gives general formula to determine the values of A_n's[2].

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\sum_{i=0}^{n} \lambda^i u_i)]_{\lambda=0}, \quad n = 0, 1, 2, \cdots \tag{2.1.10}$$

The first three term of A_n's are

$A_0 = \frac{1}{0!}\frac{d^0}{d\lambda^0}[N(\sum_{i=0}^{0} \lambda^i u_i)]_{\lambda=0} = N(u_0)$

$A_1 = \frac{1}{1!}\frac{d^1}{d\lambda^1}[N(\sum_{i=0}^{1} \lambda^i u_i)]_{\lambda=0} = \frac{d}{d\lambda}[N(\lambda^0 u_0 + \lambda^1 u_1)]_{\lambda=0}$

$= [N'(\lambda^0 u_0 + \lambda^1 u_1]_{\lambda=0}(u_1) = u_1 N'(u_0)$

$A_2 = \frac{1}{2!}\frac{d^2}{d\lambda^2}[N(\sum_{i=0}^{2} \lambda^i u_i)]_{\lambda=0} = \frac{1}{2!}\frac{d^2}{d\lambda^2}[N(\lambda^0 u_0 + \lambda^1 u_1 + \lambda^2 u_2]_{\lambda=0}$

$= \frac{1}{2!}\frac{d}{d\lambda}[N'(\lambda^0 u_0 + \lambda^1 u_1 + \lambda^2 u_2)(u_1 + 2\lambda u_2)]_{\lambda=0} = \frac{1}{2!}[N'(\lambda^0 u_0 + \lambda^1 u_1 + \lambda^2 u_2)(2u_2) + N''(\lambda^0 u_0 + \lambda^1 u_1 + \lambda^2 u_2)(u_1 + 2\lambda u_2)^2]_{\lambda=0}$

$= \frac{u_1^2}{2!}N''(u_0) + u_2 N'(u_0)$

The ADM is similar to find the Taylor's series expansion for the nonlinear function $N(u)$ around the initial function u_0.

$$N(u) = N(u_0) + N'(u_0)(u - u_0) + \frac{1}{2!}N''(u_0)(u - u_0)^2 + \cdots$$

since from ADM method $u = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \cdots$ substituting this in the above expansion we get

$$N(u) = N(u_0) + N'(u_0)(u_1 + u_2 + \cdots) + \frac{1}{2!}N''(u_0)(u_1 + u_2 + \cdots)^2 + \frac{1}{3!}N'''(u_0)(u_1 + u_2 + \cdots)^3 + \cdots$$

after that we take apart the expansion terms

$N(u) = N(u_0) + N'(u_0)(u_1) + N'u_2 + N'u_3 + \cdots + \frac{1}{2!}N''(u_0)(u_1)^2 + \frac{1}{2!}N''(u_0)u_1u_2 + \frac{1}{2!}N''(u_0)u_2u_1 + \frac{1}{2!}N''(u_0)(u_2)^2 + \frac{1}{2!}N''(u_0)u_1u_3 + \frac{1}{2!}N''(u_0)u_3u_1 + \cdots + \frac{1}{3!}N'''(u_0)(u_1)^3 + \frac{1}{3!}N'''(u_0)(u_1)^2u_2 + \frac{1}{3!}N'''(u_0)u_2(u_1)^2 + \frac{1}{3!}N'''(u_0)u_1u_2u_1 + \cdots$,

and by reordering the terms and determining the order of each term which depends on both the subscripts and the exponent of the u_n's. For example the order of $u_m^n$is mn for example u_1^2 is of order $1 \times 2 = 2$ and the order of $u_m u_n$ is $m+n$. For example u_0u_1 is of order $0 + 1 = 1$ and the order of $u_n^m u_k^l$ is $mn + kl$, for example $u_2^3 u_1^3$ is of order $(2 \times 3) + (1 \times 3) = 6 + 3 = 9$ and so on. Therefor, we get

$N(u) = N(u_0) + N\prime(u_0)u_1 + N'(u_0)u_2 + \frac{1}{2!}N''(u_0)u_1^2 + N'(u_0)u_3 + \frac{2}{2!}N''(u_0)u_1u_2 + \frac{1}{3!}N'''(u_0)u_1^3 + N'(u_0)u_4 + \frac{1}{2!}N''(u_0)u_2^2 + \frac{2}{2!}N''(u_0)u_1u_3 + \frac{3}{3!}N'''(u_0)u_1^2u_2 + \cdots$

By comparing the terms from the previous formula with the terms of the assumption $N(u) = \sum_{n=0}^{\infty} A_n$ the values of A_n's can be constructed as follow

$A_0 = N(u_0)$

$A_1 = u_1 N'(u_0)$

$A_2 = u_2 N'(u_0) + \frac{u_1^2}{2!}N''(u_0)$

$A_3 = u_3 N'(u_0) + \frac{2u_1 u_2}{2!} N''(u_0) + \frac{u_1^3}{3!} N'''(u_0)$

$$\vdots$$

which are the same values that we got from the Adomian's general formula (2.1.10) used to determine the Adomian polynomials A_n.

2.2 The Convergence Analysis of ADM

Cherruault has given the first proof of convergence of the ADM and he used fixed point theorems for abstract functional equations. In this section we give the proof of convergence of the Adomian decomposition method[1, 16]

Consider the general functional equation

$$u - Nu = f, \qquad u \in H \tag{2.2.1}$$

where H is the Hilbert space and N is the nonlinear operator $N : H \to H$ and $f = L^{-1}g$ is also in H. From the last section the ADM is based on assuming that the solution u and the nonlinear function $N(u)$ are decomposed into infinite series $u = \sum_{n=0}^{\infty} u_n$ and $N(u) = \sum_{n=0}^{\infty} A_n$.

Substituting these decomposition series in (2.2.1) yields

$$\sum_{n=0}^{\infty} u_n - \sum_{n=0}^{\infty} A_n = f$$

then the recursive terms are got from this algorithm

$$u_0 = f$$

$$u_{n+1} = A_n(u_0, u_1, \ldots, u_n)$$

The Adomian decomposition method is equivalent to find the sequence

$S_n = u_1 + u_2 + u_3 + \cdots + u_n$ by using iterative scheme

$S_0 = 0$

$S_{n+1} = N(S_n + u_0)$,

where $N(S_n + u_0) = \sum_{k=0}^{n} A_k$,

If this limit exist

$$S = \lim_{n\to\infty} S_n$$

in a Hilbert space, then S is a solution of the fixed point functional equation $S = N(u_0 + S)$ in H.

Theorem 2.1. *[2] Let N be a nonlinear operator from a Hilbert space H where $N : H \to H$ and u be the exact solution of (2.2.1). The decomposition series $\sum_{n=0}^{\infty} u_n$ of u converges to u when*

$$\exists\alpha < 1, \| u_{n+1} \| \leq \alpha \| u_n \|, \forall n \in N \cup \{0\}.$$

Proof. *We have the sequence*

$$S_n = u_1 + u_2 + \cdots + u_n$$

We need to show that this sequence is a Cauchy sequence in the Hilbert space H.

$$\| S_{n+1} - S_n \| = \| u_{n+1} \| \leq \alpha \| u_n \| \leq \alpha^2 \| u_{n-1} \| \leq \cdots \leq \alpha^{n+1} \| u_0 \|$$

In order to prove that S_n is Cauchy sequence

$$\begin{aligned}
\| S_m - S_n \| &= \| (S_m - S_{m-1}) + (S_{m-1} - S_{m-2}) + \cdots + (S_{n+1} - S_n) \| \\
&\leq \| S_m - S_{m-1} \| + \| S_{m-1} - S_{m-2} \| + \| S_{m-2} - S_{m-3} \| + \cdots + \| S_{n+1} - S_n \| \\
&\leq \alpha^m \| u_0 \| + \alpha^{m-1} \| u_0 \| + \cdots + \alpha^{n+1} \| u_0 \| \\
&= (\alpha^m + \alpha^{m-1} + \cdots + \alpha^{n+1}) \| u_0 \| \\
&\leq (\alpha^{n+1} + \alpha^{n+2} + \cdots) \| u_0 \|
\end{aligned}$$

then,

$$\| S_m - S_n \| = \frac{\alpha^{n+1}}{1-\alpha} \| u_0 \|, \ for \ n, m \in N, m \geq n \tag{2.2.2}$$

Since $\alpha < 1$. *From (2.1), the sequence* $\{S_n\}_{n=0}^{\infty}$*is a Cauchy sequence in the Hilbert space. Hence,*

$$lim_{n\to\infty} S_n = S, forS \in H$$

where $S = \sum_{n=0}^{\infty} u$. *Solving (2.2.1) is the same as solving the functional equation* $N(S + u_0) = S$; *by assuming that* N *is a continuous operator we get*

$$N(S + u_0) = N(\lim_{n\to\infty}(S_n + u_0)) = \lim_{n\to\infty} N(S_n + u_0)) = \lim S_{n+1} = S$$

so S *is the solution of (2.2.1).*

2.2.1 The Convergence order of ADM

The order of convergence of the ADM was discussed by Babolian and Biazer[11].

Definition 2.1. *[11] Let* S_n *be a sequence that converges to* S. *If there exist two constants* p *and* c, $c \in \mathbb{R}$, $p \in \mathbb{N}$, *such that*

$$\lim_{n\to\infty} \mid \frac{S_{n+1} - S}{(S_n - S)^p} \mid = c \tag{2.2.3}$$

then the order of convergence of S_n *is* p.

To determine the order of convergence of S_n, consider the Taylor expansion of $N(S_n + u_o)$ around the point $(S + u_o)$:

$$N(S_n + u_o) = N(S + u_o) + N'(S + u_o)(S_n - S) + \frac{1}{2!}N''(S + u_o)(S_n - S)^2 + \cdots$$

$$+\frac{1}{m!}N^m(S+u_o)(S_n-S)^m+\cdots$$

$$N(S_n+u_o)-N(S+u_o)=N'(S+u_o)(S_n-S)+\frac{1}{2!}N''(S+u_o)(S_n-S)^2+\cdots$$

$$+\frac{1}{m!}N^m(S+u_o)(S_n-S)^m+\cdots \tag{2.2.4}$$

Since $N(S+u_o)=S$ and $N(S_n+u_o)=S_{n+1}$, so (2.2.4) becomes

$$S_{n+1}-S=N'(S+u_o)(S_n-S)+\frac{1}{2!}N''(S+u_o)(S_n-S)^2+\cdots$$
$$+\frac{1}{m!}N^m(S+u_o)(S_n-S)^m+\cdots \tag{2.2.5}$$

Theorem 2.2. *[11] Suppose $N\in C^p[a,b]$ if $N^m(S+u_o)=0$ for $m=0,1,2,\ldots,p-1$ and $N^p(S+u_o)\neq 0$, then the sequence S_n is of order p.*

Proof. *By the hypotheses of theorem, from (2.2.5) we have:*

$$S_{n+1}-S=\frac{1}{p!}N^p(S+u_o)(S_n-S)^p+\frac{1}{p+1!}N^{p+1}(S+u_o)(S_n-S)^{p+1}+\cdots \tag{2.2.6}$$

By dividing both sides of this equation by (S_n-S) we get

$$\frac{S_{n+1}-S}{(S_n-S)^p}=\frac{1}{p!}N^p(S+u_o)+\frac{1}{p+1!}N^{p+1}(S+u_o)(S_n-S)+\cdots \tag{2.2.7}$$

Then we take the limit as $n\to\infty$ to both sides of equation (2.3.1)

$$\lim_{n\to\infty}\left|\frac{S_{n+1}-S}{(S_n-S)^p}\right|=\lim_{n\to\infty}\frac{1}{p!}N^p(S+u_o)+\lim_{n\to\infty}\frac{1}{p+1!}N^{p+1}(S+u_o)(S_n-S)+\cdots \tag{2.2.8}$$

Since $lim_{n\to\infty}(S_n)=S$ then every terms that has (S_n-S) will be canceled so at the end we have

$$\lim_{n\to\infty}\left|\frac{S_{n+1}-S}{(S_n-S)^p}\right|=\lim_{n\to\infty}\frac{1}{p!}N^p(S+u_o)=c \tag{2.2.9}$$

so by definition 2.1 the order of the sequence is p.

2.3 Some Modifications Of ADM

In this section, some modifications of ADM are presented [34, ?].

2.3.1 Modified Adomian Method

Power series solutions of linear homogeneous differential equations yield simple recurrence relations for the coefficients, but they are not suitable for nonlinear equations in general. Consider the result from [8, 34]

$$N(\sum_{n=0}^{\infty} c_n x^n) = \sum_{n=0}^{\infty} x^n A_n(c_0, c_1, \ldots, c_n)$$

from the recent theorem of Adomian and Rach on transformation of series and A_n are Adomian polynomials. Since ADM gives solutions of the general equation $u - Nu = f$ using his decompositions $u = \sum_{n=0}^{\infty} u_n$, and $N(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n)$. If u is given as power series $u = \sum_{n=0}^{\infty} c_n x^n$, by identifying each component u_n of u with the component $c_n x^n$ of the power series, this gives

$$A_n(u_0, u_1, u_2, \ldots, u_n) = x^n A_n(c_0, c_1, c_2, \ldots, c_n) \tag{2.3.1}$$

If we return to the formula(2.1.10) and finding the Adomian polynomials by substituting each component u_n with the component $c_n x^n$ of the power series we get

$A_0(u_0) = N(u_0) = N(c_0 x^0) = N(c_0) = A_0(c_0)$

$A_1(u_0, u_1) = u_1 N'(u_0) = c_1 x^1 N'(c_0 x^0) = x^1 c_1 N'(c_0) = x A_1(c_0, c_1)$

$A_2(u_0, u_1, u_2) = u_2 N'(u_0) + \frac{u_1^2}{2!} N''(u_0) = c_2 x^2 N'(c_0 x^0) + \frac{(c_1 x^1)^2}{2!} N''(c_0 x^0)$

$= x^2 c_2 N'(c_0) + x^2 \frac{(c_1)^2}{2!} N''(c_0) = x^2 (c_2 N'(c_0) + \frac{(c_1)^2}{2!} N''(c_0)) = x^2 A_2(c_0, c_1, c_2)$

$A_3(u_0, u_1, u_2, u_3) = u_3 N'(u_0) + \frac{2 u_1 u_2}{2!} N''(u_0) + \frac{u_1^3}{3!} N'''(u_0)$

$= c_3 x^3 N'(c_0 x^0) + \frac{2(c_1 x^1)(c_2 x^2)}{2!} N''(c_0 x^0) + \frac{(c_1 x^1)^3}{3!} N'''(c_0 x^0) = x^3 c_3 N'(c_0) + \frac{2(c_1 x)(c_2 x^2)}{2!} N''(c_0) +$

$\frac{(c_1 x)^3}{3!} N'''(c_0) = x^3[c_3 N'(c_0) + \frac{2c_1c_2}{2!} N''(c_0) + \frac{(c_1)^3}{3!} N'''(c_0)] = x^3 A_3(c_0, c_1, c_2, c_3)$

$$\vdots$$

then by mathematical induction we can get relation (2.3.1).
If the series $\sum_{n=0}^{\infty} c_n x^n$ is convergent then the series $\sum_{n=0}^{\infty} x^n A_n(c_1, c_2, \ldots, c_n)$ is convergent.
We will show how can we apply this modification in example at the end of this section.

2.3.2 Wazwaz Modifications

Another modifications to ADM was proposed by Wazwaz[40, 8, 9].

The New Modification

In the new modification Wazwaz replace $f = L^{-1}g$ by a series of infinite components.

$$f = \sum_{n=0}^{\infty} f_n.$$

So the new recursive relationship is presented in the form

$$u_0 = f_0$$

$$u_{n+1} = f_n - L^{-1}Ru_n - L^{-1}(A_n), \quad for \;\; n = 0, \; 1, \; 2, \cdots$$

The benefit of this method is the size of calculations is minimized compared to standard ADM and this reduction facilitates the construction of Adomian polynomials for nonlinear operators.

Reliable Modification

This method is based on splitting f into two parts $f = f_0 + f_1$. Consequently, the recursive relation

$$u_0 = f_0$$

$$u_1 = f_1 - L^{-1}Ru_0 - L^{-1}A_0$$

$$u_{n+2} = -L^{-1}Ru_{n+1} - L^{-1}A_{n+1}, \quad for \;\; n = 0,\ 1,\ 2, \cdots$$

Since this variation is not very large but it plays a major role in accelerating the convergence of the solution and it minimize the size of calculations. The success of this modification depends on the choice of f_0 and f_1, and this come from trials.

2.3.3 Two-Step Adomian Method

The main ideas of Two Step Adomian Method ($TSADM$) was discussed in [40, 8, 9] and the two steps are written below:
Step1: Using the given conditions, we obtain

$$\Phi = \phi + L^{-1}g$$

where the function ϕ represents the terms arising from using the given conditions, all are assumed to be prescribed. We set

$$\Phi = \Phi_0 + \Phi_1 + \cdots + \Phi_m$$

where $\Phi_0, \Phi_1, \ldots, \Phi_m$are the terms arising from integrating the source term g. We define

$$u_0 = \Phi_k + \cdots + \Phi_{k+s}$$

where $k = 0, 1, \ldots, m,\ s = 0, 1, \ldots, m - k$. Then we verify that u_0 satisfies the original equation and the given conditions by substitution, once the exact solution is obtained we finish. Otherwise, we go to step two.

Step2: We set $u_0 = \Phi$ and continue with the standard Adomian recursive relation $u_{n+1} = -L^{-1}(A_n)$.

Compared to the standard Adomian method and the modified method, we can see that the two-step Adomian method may provide the solution by using one iteration only. Further, the $(TSADM)$ avoids the difficulties arising in the modified method. Furthermore, the number of terms in Φ namely m, is small in many practical problems.

The example below will be solved by the standard ADM, modified Adomian method, the new modification, reliable modification and Two-Step Adomian Method. This example will show how these modifications of the Adomian decomposition method give the exact solution with iterations than that found by using the standard method.

Example 2.3.1. *[9] Consider the equation*

$$y' - y = x\cos(x) - x\sin(x) + \sin(x), \tag{2.3.2}$$

subject to the initial condition $y(0) = 0$.

Standard ADM

In this example $L = \frac{d}{dx}$ *so rewrite (2.3.2) in operator form*

$$L(y) = Ry + g \tag{2.3.3}$$

Where $Ry = y$ *is the remainder linear term and* $g = xcosx - xsinx + sinx$.

Applying $L^{-1} = \int_0^x (.)ds$ *to both sides of equation (2.3.3)*

$$L^{-1}L(y) = L^{-1}Ry + L^{-1}g$$

$$\int_0^x (y')ds = y(s)\mid_0^x = L^{-1}Ry + L^{-1}g$$

$$y - y(0) = L^{-1}(y) + L^{-1}(x\cos(x) - x\sin(x) + \sin(x))$$

then,

$$y - y(0) = \int_0^x y(s)ds + \int_0^x (s\cos(s) - s\sin(s) + \sin(s))ds \qquad (2.3.4)$$

to find the integral $\int_0^x (s\cos(s) - s\sin(s) + \sin(s))ds$ *we use integration by parts* $\int_0^x (s\cos(s) - s\sin(s) + sin(s))ds = s\sin(s) \mid_0^x - \cos(s) \mid_0^x + s\cos(s) \mid_0^x - \sin(s) \mid_0^x + \cos(s) \mid_0^x = x\sin(x) - \cos(x) + 1 + x\cos(x) - \sin(x) + \cos(x) - 1$ $= x\sin(x) + x\cos(x) - \sin(x)$, *and substituting it in (2.3.4) to get*

$$y = L^{-1}(y) + x\sin(s) + x\cos(x) - \sin(x),$$

substituting the decomposition series $y(x) = \sum_{n=0}^{\infty} y_n(x)$ *in equation above*

$$\sum_{n=0}^{\infty} y_n(x) = L^{-1}(\sum_{n=0}^{\infty} y_n(x)) + x\sin(x) + x\cos(x) - \sin(x)$$

Thus the recursive relationship is given as follows

$$y_0 = x\sin(s) + x\cos(x) - \sin(x)$$

$$y_{n+1} = L^{-1}(y_n)$$

Then the first three terms are(all the following integration are found by using integration by parts):

$$y_1 = L^{-1}(y_0) = \int_0^x (s\sin(s) + s\cos(s) - \sin(s))ds$$
$$= -x\cos(x) + \sin(x) + x\sin(x) + 2\cos(x) - 2$$
$$y_2 = L^{-1}(y_1) = \int_0^x (-s\cos(s) + \sin(s) + s\sin(s) + 2\cos(s) - 2)ds$$
$$= -x\sin(x) - 2\cos(x) + 3\sin(x) - x\cos(x) - 2x + 2$$
$$y_3 = L^{-1}(y_2) = \int_0^x (-s\sin(s) - 2\cos(s) + 3\sin(s) - s\cos(s) - 2s + 2)ds$$
$$= x\cos(x) - 3\sin(x) - x\sin(x) - \cos(x) + 2x - x^2 + 1$$

thus,

$y = y_0 + y_1 + y_2 + y_3 + \cdots$

$= x\sin(x) + x\cos(x) - \sin(x) - x\cos(x) + \sin(x) + x\sin(x) + 2\cos(x) - 2 - x\sin(x) - 2\cos(x) + 3\sin(x) - x\cos(x) - 2x + 2 + x\cos(x) - 3\ sin(x) - x\sin(x) - \cos(x) + 2x - x^2 + 1 + \cdots$

after many iterations, $y = x\sin(x)$ *will be the solution of example (2.3.1) and the other terms are canceled with each other.*

By Modified Adomian Method

Assume that $y_n = \sum_{n=0}^{\infty} c_n x^n$ *and* $g = xcosx - xsinx + sinx = \sum_{n=0}^{\infty} b_n x^n$*. Substituting these values in (2.3.2)*

$$\sum_{n=0}^{\infty} c_n x^n = y(0) + L^{-1}(\sum_{n=0}^{\infty} c_n x^n) + L^{-1}(\sum_{n=0}^{\infty} b_n x^n). \tag{2.3.5}$$

Since $L^{-1} = \int_0^x (.)ds$*, then*

$$L^{-1}(\sum_{n=0}^{\infty} c_n x^n)ds = \sum_{n=0}^{\infty} \int_0^x (c_n s^n)ds = \sum_{n=0}^{\infty} c_n \frac{x^{n+1}}{n+1}.$$

Therefor, equation(2.3.5) *is given by*

$$\sum_{n=0}^{\infty} c_n x^n = y(0) + (\sum_{n=0}^{\infty} \frac{(b_n + c_n)}{n+1} x^{n+1}),$$

then,

$$\sum_{n=0}^{\infty} c_n x^n = y(0) + (\sum_{n=1}^{\infty} \frac{(b_{n-1} + c_{n-1})}{n} x^n)$$

the recursive terms are given by

$c_0 = y(0) = 0$

$c_{n+1} = \frac{b_{n-1} + c_{n-1}}{n}.$

Taylor series expansion of g is

$$g = x\cos(x) - x\sin(x) + \sin(x) = x(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots) - x(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots) + (x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots) = 2x - x^2 - \frac{4x^3}{3!} - \frac{x^4}{3!} + \frac{6x^5}{5!} - \frac{x^6}{5!} + \cdots.$$

By considering the assumption $g = \sum_{n=1}^{\infty} b_n x^n$ *and by matching the terms of this assumption with the opposite terms from above equation we get*

$$b_0 = 0$$

$$b_1 = 1$$

$$b_2 = -1$$

$$b_3 = \frac{-4}{3!}$$

$$b_4 = \frac{-1}{6}$$

$$b_5 = \frac{6}{5!}$$

$$\vdots$$

so to find the terms of $y(x)$*, we need to determine the* c_n*'s*

$$c_1 = \frac{b_0 + c_0}{1} = \frac{0}{1} = 0$$

$$c_2 = \frac{b_1 + c_1}{2} = \frac{2+0}{2} = 1$$

$$c_3 = \frac{b_2 + c_2}{3} = \frac{-1+1}{3} = 0$$

$$c_4 = \frac{b_3 + c_3}{4} = \frac{-4/3!}{4} = -\frac{1}{3!}$$

$$c_5 = \frac{b_4 + c_4}{5} = \frac{-1/6 + 1/6}{5} = 0$$

$$c_6 = \frac{b_5 + c_5}{6} = \frac{6/5! + 0}{6} = \frac{1}{5!}$$

$$\vdots$$

the exact solution is

$y(x) = \sum_{n=1}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + \cdots$
$= 0 + 0 + x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} + \cdots = x(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots) = xsinx$

By Reliable Modification

let $f_1 = x\sin(x)$ *and* $f_2 = x\cos(x) - \sin(x)$, *then the first two terms of* y_n*'s are given by*

$$y_0 = f_1 = x\sin(x)$$

$$y_1 = f_2 + L^{-1}(y_0) = x\cos(x) - \sin(x) + \int_0^x (s\sin(s))ds = x\cos(x) - \sin(x) - x\cos(x) + \sin(x) = 0$$

Then $y_{n+1} = 0$ *for* $n = 1, 2, 3, \ldots,$ *so the exact solution is* $y(x) = x\sin(x)$

By The New Modification

Since $f(x) = x\sin(x) + x\cos(x) - \sin(x)$, *Taylor series of* $f(x)$ *is*

$$f(x) = x^2 - \frac{2x^3}{3!} - \frac{x^4}{3!} + \frac{4x^5}{5!} + \frac{x^6}{5!} - \cdots$$

so the recursive relationship is

$$\begin{aligned}
y_0 &= f_0(x) = x^2 \\
y_1 &= f_1(x) + L^{-1}y_0 = -\frac{2x^3}{3!} + \int_0^x (s^2)ds \\
&= -\frac{2x^3}{3!} + \frac{x^3}{3} = 0 \\
y_2 &= f_2(x) + L^{-1}y_1 = -\frac{x^4}{3!} + \int_0^x (0)ds \\
&= -\frac{x^4}{3!} \\
y_3 &= f_3(x) + L^{-1}y_2 = \frac{4x^5}{5!} + \int_0^x (s^4/3!)ds \\
&= \frac{4x^5}{5!} + \frac{x^5}{5*3!} = 0 \\
&\vdots
\end{aligned}$$

so

$$y(x) = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \cdots = x(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots) = x\sin(x)$$

By TSMADM

From the integration of $g(x) = x\cos(x) - x\sin(x) + \sin(x)$ *we get* $\phi = x\sin(x) +$

$x\cos(x) - \sin(x)$ *so let* $\phi = \phi_0 + \phi_1 + \phi_2$ *where* $\phi_0 = x\sin(x)$, $\phi_1 = x\cos(x)$ *and* $\phi_2 = -\sin(x)$, *each of them satisfies the initial condition* $y(0) = 0$ *so if we choose* $\phi_0 = y_0(x)$ *we get the exact solution because* $y(x) = x\sin(x)$ *satisfies the initial value problem described in example*(2.3.1).

CHAPTER 3

Applications on ADM

The ADM gives the accurate and efficient solution for wide class of linear and nonlinear equations without the need to resort to linearization or perturbation approaches. In this chapter we will show firstly how ADM can apply to solve linear and nonlinear initial value problems of ordinary differential equations, boundary value ordinary differential equations and system of ordinary differential equations. Then we will use the ADM to find the solution of partial differential equations of first order, second order especially heat and wave equations and equations of higher order. At the end of this chapter, we will give examples of solving integral equations with ADM.

3.1 Ordinary Differential Equation

In this section we will apply the ADM to linear and nonlinear ordinary differential equations[2]. Consider the nonlinear **first order** initial value differential equation

$$u' + Nu + Ru = g$$
$$u(x_0) = c_0 \tag{3.1.1}$$

Where N is the nonlinear term, R is the remainder linear term and g is a given function. In this case$L = \frac{d}{dx}$ so $L^{-1} = \int_{x_0}^{x} (.)ds$, for (3.1.1) take L^{-1} to both sides

$$L^{-1}Lu = L^{-1}g - L^{-1}Nu - L^{-1}Ru$$
$$\int_{x_0}^{x} (u')ds = L^{-1}g - L^{-1}Nu - L^{-1}Ru$$
$$u(x) - u(x_0) = L^{-1}g - L^{-1}Nu - L^{-1}Ru$$

$$u(x) = u(x_0) + L^{-1}g - L^{-1}Nu - L^{-1}Ru \tag{3.1.2}$$

The ADM gives the solution u as an infinite series $u(x) = \sum_{n=0}^{\infty} u_n(x)$ and the nonlinear term $Nu = \sum_{n=0}^{\infty} A_n$ where A_n's is the Adomian polynomials. So (3.1.2) become

$$\sum_{n=0}^{\infty} u_n = u(x_0) + L^{-1}g - L^{-1}\sum_{n=0}^{\infty} A_n - L^{-1}\sum_{n=0}^{\infty} Ru_n$$

From this equation we can get an algorithm to find the values of u_n's as follows

$$u_0 = u(x_0) + L^{-1}g$$

$$u_{n+1} = -L^{-1}(A_n + Ru_n), \quad n = 0,\ 1,\ 2, \cdots$$

3.1.1 Second Order initial value Ordinary differential equation

In this section we will use the ADM to solve initial value ordinary differential equation of second order and then we will give generalization to solve initial value ordinary differential equations of any order [2].
Consider the initial value differential equation in Adomian method operator

$$Lu + Nu + Ru = g$$
$$u(x_0) = c_0, \quad u'(x_0) = c_1. \tag{3.1.3}$$

Where $L = \frac{d^2}{dx^2}$ and so $L^{-1} = \int_{x_0}^{x}\int_{x_0}^{x}(.)dsds$. By applying L^{-1} to both sides of (3.1.3) we have

$$L^{-1}Lu(x) = \int_{x_0}^{x}\int_{x_0}^{x}(u''(s))dsds = \int_{x_0}^{x}(u'(s) - u'(x_0))ds = u(s)\mid_{x_0}^{x} - u'(x_0)(s)\mid_{x_0}^{x}$$
$$= u(x) - u(x_0) - u'(x_0)(x - x_0) = L^{-1}g - L^{-1}Nu - L^{-1}Ru$$

then,

$$u(x) = u(x_0) + u'(x_0)(x - x_0) + L^{-1}g - L^{-1}Nu - L^{-1}Ru \tag{3.1.4}$$

Substituting $u(x) = \sum_{n=0}^{\infty} u_n(x)$ and $Nu = \sum_{n=0}^{\infty} A_n$ in (3.1.4) to obtain

$$\sum_{n=0}^{\infty} u_n = u(x_0) + u'(x_0)(x - x_0) + L^{-1}g - L^{-1}\sum_{n=0}^{\infty} A_n - L^{-1}\sum_{n=0}^{\infty} Ru_n$$

Then the u_n's terms can be found by using the following recursive relationship

$$u_0 = u(x_0) + u^{'}(x_0)(x - x_0) + L^{-1}g$$
$$u_{n+1} = -L^{-1}\sum_{n=0}^{\infty} A_n - L^{-1}\sum_{n=0}^{\infty} Ru_n$$

Example 3.1.1. *(Anharmonic Oscillator)[3] Consider this equation*

$$\frac{d^2\theta}{dt^2} + k^2 sin\theta = 0 \tag{3.1.5}$$

with $k^2 = \omega/l$ *where* ω *is the angular frequency and* l *is the largest amplitude of motion. Assuming* $\theta(0) = c$ *and* $\theta'(0) = 0$ *so* $(ref26)$ *in operator form is*

$$L\theta + N\theta = 0 \tag{3.1.6}$$

where $L = \frac{d^2}{dt^2}$ *and* $N\theta = k^2 sin\theta$ *so* $L^{-1} = \int_0^t \int_0^t (.)dsds$. *Apply* L^{-1} *to* (3.1.6) *we have*

$$\begin{aligned} L^{-1}L\theta &= \int_0^t \int_0^t \theta''(\tau)d\tau d\tau \\ &= \int_0^t \theta'(\tau) \mid_0^t d\tau \\ &= \int_0^t (\theta'(\tau) - \theta'(0))d\tau \\ &= \theta(\tau) \mid_0^t - \theta'(0)\tau \mid_0^t L^{-1}L\theta + L^{-1}N\theta \quad = \theta(t) + \theta(0) + \theta'(0)(t) + L^{-1}N\theta = 0 \end{aligned}$$

so, we get

$$\theta(t) = \theta(0) + \theta'(0)(t) - L^{-1}N\theta \tag{3.1.7}$$

Adomian decomposition method assume $\theta(t)$ *is decomposed in infinite series* $\theta(t) = \sum_{n=0}^{\infty} \theta_n(t)$ *and* $N\theta = \sum_{n=0}^{\infty} A_n$. *By substituting these series and the initial conditions* $\theta(0) = c$ *and* $\theta'(0) = 0$ *in* (3.1.7) *we have*

$$\sum_{n=0}^{\infty} \theta_n = c - L^{-1} \sum_{n=0}^{\infty} A_n$$

define $\theta_0(t) = c$, *so the other terms are*

$$\theta_{n+1}(t) = -L^{-1}A_n, n = 0,\ 1,\ 2, \ldots \tag{3.1.8}$$

To get the values of A_n*'s we use the Adomian's formula* (2.1.10) *then we can find* θ_n *terms from* 3.1.8. *The first three terms of* A_n*'s and* u_n*'s are*

$A_0 = N(\theta_0) = N(c) = k^2 sin(c)$

$\theta_1 = -L^{-1}A_0 = -\int_0^t \int_0^t (k^2 sin(c))dsds = -k^2 sin(c)\frac{t^2}{2}$

$A_1 = \frac{1}{1!}\frac{d}{d\lambda}[N(\theta_0 + \theta_1\lambda)]_{\lambda=0}$

$= \frac{d}{d\lambda}[k^2 sin(\theta_0 + \theta_1\lambda)]_{\lambda=0}$

$= \frac{d}{d\lambda}[k^2 sin(c - k^2 sin(c)(t^2/2)\lambda)]_{\lambda=0}$

$= k^2 cos(c - k^2\frac{t^2}{2}sin(c)\lambda) * (-k^2 sin(c)\frac{t^2}{2}) \mid_{\lambda=0}$

$= -k^4 cos(c)sin(c)\frac{t^2}{2}$

$\theta_2 = -L^{-1}A_1 = -\int_0^t \int_0^t (-k^4 cos(c)sin(c)\frac{s^2}{2})dsds = k^4 cos(c)sin(c)\frac{t^4}{24}$

$A_2 = \frac{1}{2!}\frac{d^2}{d\lambda^2}[N(\theta_0 + \theta_1\lambda + \theta_2\lambda^2)]_{\lambda=0}$

$= \frac{1}{2!}\frac{d^2}{d\lambda^2}[k^2 sin(c - k^2 sin(c)\frac{t^2}{2}\lambda + k^4 cos(c)sin(c)\frac{t^4}{24}\lambda^2)]_{\lambda=0}$

$= \frac{1}{2!}\frac{d}{d\lambda}[k^2 con(c - k^2 sin(c)(t^2/2)\lambda + k^4 cos(c)sin(c)\frac{t^4}{24}\lambda^2)(-k^2 sin(c)\frac{t^2}{2}$

$+ 2k^4 cos(c)sin(c)\frac{t^4}{24}\lambda)]_{\lambda=0}$

$= \frac{1}{2!}[-k^2 sin(c)\frac{t^2}{2} * -k^2 sin(c - k^2\frac{t^2}{2}sin(c)\lambda + 2k^4 cos(c)sin(c)\frac{t^4}{24}\lambda) * (-k^2 sin(c)\frac{t^2}{2} +$
$2k^4 cos(c)sin(c)\frac{t^4}{24}\lambda) + k^2 cos(c - k^2 sin(c)\frac{t^2}{2}\lambda + k^4 cos(c)sin(c)\frac{t^4}{24}\lambda) * 2t^4 cos(c)sin(c)\frac{t^4}{24} +$
$2k^4 cos(c)sin(c)\frac{t^4}{24}\lambda * -k^2 sin(c - k^2 sin(c)\frac{t^2}{2}\lambda + k^4 cos(c)sin(c)\frac{t^4}{24}\lambda) * ((-k^2 cos(c)\frac{t^2}{2} +$
$2k^4 cos(c)sin(c)\frac{t^4}{24}\lambda) \mid_{\lambda=0}$

$= \frac{1}{2!}[-k^6 sin^3\frac{t^4}{4} + 2k^6 cos^2(c)sin(c)\frac{t^4}{24}$

$\theta_3 = -L^{-1}A_2 = -\int_0^t \int_0^t (-k^6 sin^3(c)\frac{s^4}{8} + k^6 cos^2(c)sin(c)\frac{s^4}{24})dsds$

$= 3k^6 sin^3(c)\frac{t^6}{720} - k^6 cos^2(c)sin(c)\frac{t^6}{720}$

$= 3k^6 sin^3(c)\frac{t^6}{6!} - k^6 cos^2(c)sin(c)\frac{t^6}{6!}$

$thus\ \theta = \theta_1 + \theta_2 + \theta_3 + \cdots$

Generalization:

Consider the initial value problem of order n in Adomian method operator form

$$Lu + Ru + Nu = g \tag{3.1.9}$$

with initial conditions

$$u(0) = c_0$$

$$u'(0) = c_1$$

$$u''(0) = c_2$$

$$u'''(0) = c_3$$

$$\vdots$$

$$u^{(n-1)}(0) = c_{n-1}$$

where $L = \frac{d^n}{dx^n}$ is the linear operator of order n, N is the nonlinear term, R is the remainder linear term and g is the source function of x. The inverse operator is given by

$$L^{-1} = \int_0^x \int_0^x \int_0^x \cdots \int_0^x (.)dsdsds \cdots ds$$

Take L^{-1} to equation(3.1.9) in order to get

$$u - \phi = -L^{-1}(Ru + Nu) + L^{-1}g \tag{3.1.10}$$

where ϕ is determined from the initial conditions

$$\phi = \begin{cases} u(0), & L = \frac{d}{dx}; \\ u(0) + xu'(0), & L = \frac{d^2}{dx^2}; \\ u(0) + xu'(0) + \frac{x^2}{2!}u''(0), & L = \frac{d^3}{dx^3}; \\ \vdots & \\ u(0) + xu'(0) + \frac{x^2}{2!}u''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}u^{(n-1)}(0), & L = \frac{d^{n+1}}{dx^{n+1}}. \end{cases}$$

From the algorithm of ADM $u = \sum_{n=0}^{\infty} u_n$ and $Nu = \sum_{n=0}^{\infty} A_n$. Eq (3.1.10) becomes

$$\sum_{n=0}^{\infty} u_n = \phi + L^{-1}g - L^{-1}(R\sum_{n=0}^{\infty} u_n) - L^{-1}(\sum_{n=0}^{\infty} A_n)$$

The general algorithm to find the solution of initial value problems with any order by using ADM is

$$u_0 = \phi + L^{-1}g,$$

$$u_{n+1} = -L^{-1}(Ru_n) - L^{-1}(A_n). \qquad n = 0,\ 1,\ 2, \cdots$$

3.1.2 Second order Singular Initial Value Problem

In this section an efficient modification of ADM is introduced for solving initial value problem in the second order ordinary differential equations[21]. Consider the general equation of second order singular initial value problem

$$y'' + \frac{2n}{x}y' + \frac{n(n-1)}{x^2}y + N(x,y) = g(x). \qquad n = 0,\ 1,\ 2, \cdots \tag{3.1.11}$$

with initial conditions

$$y(0) = \alpha, \qquad y'(0) = \beta$$

where $N(x,y)$ is nonlinear function, $g(x)$ is given function and α and β are real constants. In this problem we use the new definition of the differential operator

$$L = x^{-n}\frac{d^2}{dx^2}x^{-n}y \tag{3.1.12}$$

and its inverse is

$$L^{-1} = x^{-n}\int_0^x \int_0^x s^n(.)dsds$$

If we apply the differential operator defined in (3.1.12) to the function y we get the left hand side of (3.1.11)

$$\begin{aligned} Ly &= x^{-n}\frac{d^2}{dx^2}x^{-n}y = x^{-n}\frac{d}{dx}(nx^{n-1}y + x^n y') \\ &= x^{-n}(nx^{n-1}y' + n(n-1)x^{n-2}y + nx^{n-1}y' + x^n y'') \\ &= \frac{n}{x}y' + \frac{n(n-1)}{x^2}y + \frac{n}{x}y' + y'' \\ &= y'' + \frac{2n}{x}y' + \frac{n(n-1)}{x^2}y \end{aligned}$$

so (3.1.11) in Adomian operator form is

$$Ly = g(x) - N(x,y)$$

Take L^{-1} to both sides of equation above

$$L^{-1}Ly = L^{-1}(y'' + \frac{2n}{x}y' + \frac{n(n-1)}{x^2}y) = L^{-1}g(x) - L^{-1}N(x,y) \tag{3.1.13}$$

the left hand side of this equation is

$$\begin{aligned} L^{-1}(y'' + \frac{2n}{x}y' + \frac{n(n-1)}{x^2}y) &= x^{-n}\int_0^x\int_0^x s^n(y'' + \frac{2n}{s}y' + \frac{n(n-1)}{s^2}y)dsds \\ &= x^{-n}\int_0^x\int_0^x (s^n y'' + 2ns^{n-1}y' + n(n-1)s^{n-2}y)dsds \end{aligned}$$

using integration by parts to find $\int_0^x (s^n y'')ds$ let

$$u = s^n \qquad dv = y''$$
$$du = ns^{n-1} \qquad v = y'$$

so $\int_0^x (s^n y'') = x^n y' - n\int_0^x (s^{n-1}y')ds$ and the same for $\int_0^x (s^{n-1}y')ds$
let

$$u = s^{n-1} \qquad dv = y'$$
$$du = (n-1)s^{n-2} \qquad v = y$$

so $\int_0^x (s^{n-1}y') = x^{n-1}y - (n-1)\int_0^x (s^{n-2}y)ds$, thus by substituting these integral values we get
$x^{-n}\int_0^x\int_0^x (s^n y'' + 2ns^{n-1}y' + n(n-1)s^{n-2}y)dsds$
$= x^{-n}\int_0^x [s^n y' - ns^{n-1}y + n(n-1)\int_0^x (s^{n-2}y)ds$
$+2ns^{n-1}y - 2n(n-1)\int_0^x (s^{n-2}y)ds + n(n-1)s^{n-2}y]ds$
$= x^{-n}\int_0^x (s^n y' + ns^{n-1}y)ds$
$= x^{-n}s^n y(s) \mid_0^x - n\int_0^x s^{n-1}yds + n\int_0^x s^{n-1}yds) = x^{-n}x^n[y(s)] \mid_0^x = y(x) - y(0)$
$= y(x) - \alpha$
so equation(3.1.13) becomes

$$y(x) - \alpha = L^{-1}g(x) - L^{-1}N(x,y)$$

Assume that $y(x) = \sum_{n=0}^{\infty} y_n$ and $N(x, y) = \sum_{n=0}^{\infty} A_n$ where A_n's are the Adomian's polynomials so

$$\sum_{n=0}^{\infty} y_n = \alpha + L^{-1}g(x) - L^{-1}\sum_{n=0}^{\infty} A_n$$

The recursive scheme to find the solution is

$$y_0 = \alpha + L^{-1}g(x)$$

$$y_{n+1} = -L^{-1}A_n. \qquad n = 0,\ 1,\ 2, \ldots.$$

Example 3.1.2. *Consider the nonlinear singular initial value problem*

$$y'' + \frac{2}{x}y' + y^3 = 6 + x^6 \tag{3.1.14}$$

with initial conditions

$$y(0) = 0, \qquad y'(0) = 0$$

(3.1.14) *in ADM operator form is given by*

$$Ly = g(x) - N(x.y) = (6 + x^6) - y^3 \tag{3.1.15}$$

In this example $n = 1$ so $L = x^{-1}\frac{d^2}{dx^2}xy$ and its inverse $L^{-1} = x^{-1}\int_0^x \int_0^x s(.)dsds$. Applying L^{-1} to both sides of (3.1.15) *and from the previous section*
$L^{-1}Ly = y(x) - y(0)$

$$y - y(0) = L^{-1}(6 + x^6) - L^{-1}(y^3) \tag{3.1.16}$$

the value of $L^{-1}(6 + x^6)$ is

$$\begin{aligned} L^{-1}(6 + x^6) &= x^{-1}\int_0^x \int_0^x s(6 + s^6)dsds \\ &= x^{-1}\int_0^x (3s^2 + \frac{s^8}{8})ds \\ &= x^{-1}(s^3 + \frac{s^9}{72}) \mid_0^x = x^2 + \frac{x^8}{72} \end{aligned}$$

So after substituting the values of $L^{-1}(6+x^6)$, *the initial condition* $y(0)=0$ *and using the assumptions of Adomian method* $Nu=\sum_{n=0}^{\infty} A_n$, (3.1.16) *becomes*

$$\sum_{n=0}^{\infty} y_n = x^2 + \frac{x^8}{72} - L^{-1}(\sum_{n=0}^{\infty} A_n)$$

Using Wazwaz reliable modified method of ADM we have $f_0=x^2$ *and* $f_1=\frac{x^8}{72}$ *and from the equation* (2.1.10) *we get the following solution algorithm*

$$y_0 = x^2$$

$$y_{n+1} = \frac{x^8}{72} - L^{-1}(A_n), \qquad n = 0,\ 1,\ 2, \dots$$

then the first term is

$y_1 = \frac{x^8}{72} - L^{-1}(A_0)$, *since* $A_0 = N(y_0) = y_0^3 = x^6$

and so

$L^{-1}(A_0) = x^{-1}\int_0^x\int_0^x s(s^6)dsds = x^{-1}\int_0^x \frac{s^8}{8}ds = x^{-1}(\frac{s^9}{72}) \mid_0^x = \frac{x^8}{72}$

Thus

$$y_1 = \frac{x^8}{72} - \frac{x^8}{72} = 0$$

then all recursive terms are equal zero. The solution is $y(x)=x^2$

3.1.3 Boundary Value Problems

In this section, we apply the ADM to obtain numerical solution to nonlinear boundary value problem[2].

Example 3.1.3. *[2] Consider the following nonlinear sixth order boundary value problem:*

$$u^{(6)}(x) = e^{-x}u^2(x), \qquad 0 < x < 1 \tag{3.1.17}$$

with boundary conditions

$u(0) = u''(0) = u^{(4)}(0) = 1$

$u(1) = u''(1) = u^{(4)}(1) = e$

Solution:

Rewrite (3.1.17) *in Adomian method operator form*

$$Lu = e^{-x}Nu \tag{3.1.18}$$

where $L = \frac{d^6}{dx^6}$ *and* $Nu = u^2(x)$. *Applying*

$$L^{-1} = \int_0^x \int_0^x \int_0^x \int_0^x \int_0^x \int_0^x (.)dsdsdsdsdsds$$

to both sides of (3.1.18)

$$L^{-1}Lu = L^{-1}(e^{-x}Nu)$$

$$\int_0^x \int_0^x \int_0^x \int_0^x \int_0^x \int_0^x (\frac{d^6u}{dx^6})dsdsdsdsdsds = L^{-1}(e^{-x}Nu)$$

$$\int_0^x \int_0^x \int_0^x \int_0^x \int_0^x (\frac{d^5u}{dx^5} - \frac{d^5u(0)}{dx^5})dsdsdsdsds = L^{-1}(e^{-x}Nu)$$

$$\int_0^x \int_0^x \int_0^x \int_0^x (\frac{d^4u}{dx^4} - \frac{d^4u(0)}{dx^4} - s\frac{d^5u(0)}{dx^5})dsdsdsds = L^{-1}(e^{-x}Nu)$$

$$\int_0^x \int_0^x \int_0^x (\frac{d^3u}{dx^3} - \frac{d^3u(0)}{dx^3} - s\frac{d^4u(0)}{dx^4} - \frac{s^2}{2}\frac{d^5u(0)}{dx^5})dsdsds = L^{-1}(e^{-x}Nu)$$

$$\int_0^x \int_0^x (\frac{d^2u}{dx^2} - \frac{d^2u(0)}{dx^2} - s\frac{d^3u(0)}{dx^3} - \frac{s^2}{2}\frac{d^4u(0)}{dx^4} - \frac{s^3}{6}\frac{d^5u(0)}{dx^5})dsds = L^{-1}(e^{-x}Nu)$$

$$\int_0^x (\frac{du}{dx} - \frac{du(0)}{dx} - s\frac{d^2u(0)}{dx^2} - \frac{s^2}{2}\frac{d^3u(0)}{dx^3} - \frac{s^3}{6}\frac{d^4u(0)}{dx^4} - \frac{s^4}{24}\frac{d^5u(0)}{dx^5})ds = L^{-1}(e^{-x}Nu).$$

Then,

$$u(x) - c_1 - c_2x - \frac{c_3}{2}x^2 - \frac{c_4}{6}x^3 - \frac{c_5}{24}x^4 - \frac{c_6}{120}x^5 = L^{-1}[e^{-x}u^2(x)] \tag{3.1.19}$$

where $c_1 = u(0)$, $c_2 = \frac{du(0)}{dx}$, $c_3 = \frac{d^2u(0)}{dx^2}$, $c_4 = \frac{d^3u(0)}{dx^3}$, $c_5 = \frac{d^4u(0)}{dx^4}$ *and* $c_6 = \frac{d^5u(0)}{dx^5}$.

So from the given boundary conditions for equation(3.1.17) *we can find these values*

$c_1 = c_3 = c_5 = 1$

and for the other unknown constants $c_2 = \alpha$, $c_4 = \beta$ *and* $c_6 = \gamma$. *Thus* (3.1.19) *becomes*

$$u(x) = 1 + \alpha x + \frac{1}{2}x^2 + \frac{\beta}{6}x^3 + \frac{1}{24}x^4 + \frac{\gamma}{120}x^5 + L^{-1}[e^{-x}u^2(x)] \qquad (3.1.20)$$

From ADM $u(x) = \sum_{n=0}^{\infty} u_n(x)$ *and* $Nu = \sum_{n=0}^{\infty} A_n$. *Substituting in* (3.1.20)

$$\sum_{n=0}^{\infty} u_n(x) = 1 + \alpha x + \frac{1}{2}x^2 + \frac{\beta}{6}x^3 + \frac{1}{24}x^4 + \frac{\gamma}{120}x^5 + L^{-1}[e^{-x}\sum_{n=0}^{\infty} A_n]$$

The algorithm to find u is (also by using Wazwaz reliable modified method of ADM we have $f_0 = 1$ *and* $f_1 = \alpha x + \frac{1}{2}x^2 + \frac{\beta}{6}x^3 + \frac{1}{24}x^4 + \frac{\gamma}{120}x^5$*) then*

$$\begin{aligned} u_0 &= 1 \\ u_1 &= \alpha x + \frac{1}{2}x^2 + \frac{\beta}{6}x^3 + \frac{1}{24}x^4 + \frac{\gamma}{120}x^5 + L^{-1}[e^{-x}A_0] \\ u_{n+1} &= L^{-1}[e^{-x}u_n^2(x)], \qquad n = 0,\ 1,\ 2,\ 3, \cdots \end{aligned}$$

Using the Adomian formula to obtain A_n*'s polynomials we have*

$A_0 = N(u_0) = u_0^2 = 1$

then

$$\begin{aligned} L^{-1}[e^{-x}A_0] = L^{-1}[e^{-x}] &= \int_0^x \int_0^x \int_0^x \int_0^x \int_0^x \int_0^x (e^{-s}) dsdsdsdsdsds \\ &= e^{-x} + \frac{x^5}{120} - \frac{x^4}{24} + \frac{x^3}{6} - \frac{x^2}{2} + x - 1 \end{aligned}$$

substitute this in u_1 *we get*

$$u_1 = -1 + (\alpha + 1)x + (\beta + 1)\frac{x^3}{6} + (\gamma + 1)\frac{x^5}{120} + e^{-x}$$

If we approximate the solution by using the only these two terms u_0 *and* u_1.

$$u(x) = u_0 + u_1 = (\alpha + 1)x + (\beta + 1)\frac{x^3}{6} + (\gamma + 1)\frac{x^5}{120} + e^{-x}$$

By using the Taylor expansion of $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$ *in the formula of* $u(x)$ *we have the final approximation formula of* $u(x)$

$$u(x) \cong u_0 + u_1 = 1 + \alpha x + \frac{x^2}{2!} + \beta\frac{x^3}{6} + \frac{x^4}{24} + \gamma\frac{x^5}{120} + \cdots$$

to find the values of α, β *and* γ *we use the boundary conditions at* $x = 1$ *and putting it in the formula of* $u(x)$ *then*

$u(1) = 1 + \alpha + \frac{1}{2} + (1/6)(\beta) + (1/120)\gamma + \cdots = e,$ 1

$u'(x) = \alpha + x + \frac{\beta}{2}x^2 + \frac{x^3}{6} + \frac{\gamma}{24}x^4 + \cdots = e$

$u''(x) = 1 + (\beta)x + \frac{x^2}{2} + \frac{\gamma}{6}x^3 + ...$

$\Rightarrow u''(1) = 1 + (\beta) + \frac{1}{2} + \frac{\gamma}{6} + \cdots = e,$ 2

$u'''(x) = (\beta) + x + \frac{\gamma}{2}x^2 + \cdots$

$u^{(4)}(x) = 1 + (\gamma)x + \cdots \Rightarrow u''''(1) = 1 + (\gamma) + \cdots = e,$ 3

from 3 *we can find the value of* γ

$\gamma = e - 1 = 1.71828183$

Then from 2 *we find* β

$\beta = e - 1 - \frac{1}{2} - \frac{1.71828183}{6} = 0.93190153$

finally $\alpha = e - \frac{1.71828183}{120} - 1 - 0.5 - \frac{0.93190153}{6} = 1.00697924$

Thus

$u(x) \cong u_0 + u_1 = 1 + 1.00697924x + \frac{x^2}{2!} + 0.93190153\frac{x^3}{6} + \frac{x^4}{24} + 1.71828183\frac{x^5}{120} + \cdots$

3.1.4 Singular Boundary Value Problems

In this section, we will consider differential equations which possess a singularity[2, 36].

Consider the general singular boundary value problem which has the following form

$$y^{(n+1)} + \frac{m}{x} y^{(n)} + q(x)N(y) = g(x), \quad 0 \leq x \leq b \tag{3.1.21}$$
$$y(0) = \alpha_0, \quad y'(0) = \alpha_1, \quad y''(0) = \alpha_2, \cdots,$$
$$y^{n-1}(0) = \alpha_{n-1}, y'(b) = \beta \qquad n = 0,\ 1,\ 2, \cdots$$

where $N(y)$ is the nonlinear term, $g(x)$ is a given function. In order to apply the ADM, let's first write (3.1.21) in Adomian operator form

$$Ly + q(x)N(y) = g(x) \tag{3.1.22}$$

where L is given by

$$L = x^{-1} \frac{d^n}{dx^n} x^{1+n-m} \frac{d}{dx} x^{m-n}(.) \tag{3.1.23}$$

if we apply this operator to y we get

$$\begin{aligned}
Ly &= x^{-1} \frac{d^n}{dx^n} x^{1+n-m} \frac{d}{dx}(x^{m-n} y) \\
&= x^{-1} \frac{d^n}{dx^n} x^{1+n-m} [(m-n)x^{m-n-1} y + x^{m-n} y'] \\
&= x^{-1} \frac{d^n}{dx^n} [(m-n)y + xy'] \\
&= x^{-1} \frac{d^{n-1}}{dx^{n-1}} [(m-n)y' + xy'' + y'] \\
&= x^{-1} \frac{d^{n-1}}{dx^{n-1}} [(m-n+1)y' + xy''] \\
&= x^{-1} \frac{d^{n-2}}{dx^{n-2}} [(m-n+2)y'' + xy'''] \\
&\vdots \\
&= x^{-1}[xy^{(n)} + my^{(n-1)}] = [y^{(n)} + \frac{m}{x} y^{(n-1)}]
\end{aligned}$$

The inverse of L is given by

$$L^{-1} = x^{n-m} \int_b^x s^{m-n-1} \int_0^x \int_0^x \int_0^x s(.) ds ds \cdots ds \tag{3.1.24}$$

applying L^{-1} to both sides of (3.1.22)

$$L^{-1}L(y) = y(x) - \phi = L^{-1}(g(x) - q(x)N(y)) \tag{3.1.25}$$

where ϕ can be determined from the initial conditions. Use the assumptions of ADM (3.1.25) becomes

$$\sum_{n=0}^{\infty} y_n = \phi + L^{-1}g(x) - L^{-1}[q(x)\sum_{n=0}^{\infty} A_n]$$

Thus the recursive relationship is

$$y_0(x) = \phi + L^{-1}g(x)$$

$$y_{n+1} = -L^{-1}[q(x)A_n], \quad n = 0,\ 1,\ 2,\ 3, \cdots$$

Example 3.1.4. *[2] Consider this third order singular BVP*

$$y''' - \frac{2}{x}y'' = y + y^2 + 7x^2e^x + 6xe^x - 6e^x + x^6e^{2x} \tag{3.1.26}$$
$$y(0) = y'(0) = 0, \quad y'(1) = e.$$

In this problem $n = 2$, $m = -2$, $b = 1$ *and* $N(y) = y^2$*so*

$$L = x^{-1}\frac{d^2}{dx^2}x^5\frac{d}{dx}x^{-4}(.) \tag{3.1.27}$$

and its inverse is

$$L^{-1} = x^4\int_1^x s^{-5}\int_0^x\int_0^x s(.)dsds \tag{3.1.28}$$

if we apply the differential operator L *that is defined in* (3.1.27) *to the function we* *obtain the left hand side of* (3.1.26)

$$\begin{aligned}
Ly &= x^{-1}\frac{d^2}{dx^2}x^5\frac{d}{dx}x^{-4}(y) \\
&= x^{-1}\frac{d^2}{dx^2}x^5[(-4)x^{-5}y + x^{-4}y'] \\
&= x^{-1}\frac{d^2}{dx^2}[(-4)y + xy'] \\
&= x^{-1}\frac{d}{dx}[(-4)y' + xy'' + y'] \\
&= x^{-1}\frac{d}{dx}[(-3)y' + xy''] \\
&= x^{-1}[(-2)y'' + xy'''] \\
&= y''' - \frac{2}{x}y''
\end{aligned}$$

applying L^{-1} *to the operator form of* (3.1.28)

$$L^{-1}Ly = L^{-1}(y + N(y) + 7x^2e^x + 6xe^x - 6e^x + x^6e^{2x})$$

the left hand side

$$\begin{aligned}
L^{-1}Ly &= L^{-1}(y''' - \frac{2}{x}y'') \\
&= x^4\int_1^x s^{-5}\int_0^x\int_0^x s(y''' - \frac{2}{s}y'')dsds
\end{aligned}$$

using integration by parts to the first term

$$u = s \qquad dv = y'''$$

$$du = ds \qquad v = y''$$

$\int_0^x s(y''') = sy'' \mid_0^x - y' \mid_0^x = sy'' - y' + y'(0)$

$L^{-1}Ly = x^4\int_1^x x^{-5}\int_0^x[sy'' - y' + y'(0) - 2y' + 2y'(0)]ds$

we use the same technique as above for first term

$$\begin{aligned}
L^{-1}Ly &= x^4 \int_1^x s^{-5}[sy' - y + y(0) - y + y(0) + sy'(0) - 2y' + 2y'] \\
&= x^4 \int_1^x [s^{-4}y' - 4s^{-5}y + 4y(0)s^{-5} + 3s^{-4}y'(0)] \\
&= x^4[s^{-4}y \mid_0^x + 4\int_1^x s^{-5}y - 4\int_1^x s^{-5}y - y(0)s^{-4} \mid_0^x - y'(0)s^{-3} \mid_0^x] \\
&= y - x^4 y(1) = y - 2.71828x^4
\end{aligned}$$

after substituting the initial condition. Then, we have this relation

$$y - 2.71828x^4 = L^{-1}(y + y^2 + 7x^2e^x + 6xe^x - 6e^x + x^6e^{2x})$$

Use the assumptions of the ADM

$$\sum_{n=0}^{\infty} y_n = 2.71828x^4 + L^{-1}(\sum_{n=0}^{\infty} y_n + \sum_{n=0}^{\infty} A_n + 7x^2e^x + 6xe^x - 6e^x + x^6e^{2x})$$

Thus the recursive relationship is

$$\begin{aligned}
y_0(x) &= 2.71828x^4 + L^{-1}[7x^2e^x + 6xe^x - 6e^x + x^6e^{2x}] \\
y_{n+1} &= -L^{-1}[y_n + A_n], \quad n = 0,\ 1,\ 2, \cdots
\end{aligned}$$

3.1.5 System of Ordinary Differential Equations

In this section we illustrate how the AMD is used to solve a system of first order ordinary differential equations[12].

Consider the system of equations:

$$\begin{aligned}
u_1' + N_1(x, u_1, u_2, \cdots, u_n) &= g_1 \\
u_2' + N_2(x, u_1, u_2, \cdots, u_n) &= g_2 \\
&\vdots \\
u_n' + N_n(x, u_1, u_2, \cdots, u_n) &= g_n \qquad (3.1.29)
\end{aligned}$$

where $N_1, N_2, \ldots, N_n$are nonlinear functions, $g_1, g_2, \ldots, g_n$ are known functions. Rewrite (3.1.29) in operation form:

$$Lu_i + N_i(x, u_1, u_2, \ldots, u_n) = g_i, \qquad i = 1,\ 2, \cdots,\ n \tag{3.1.30}$$

Where L is the linear operator $L = \frac{d}{dx}$ and has an inverse $L^{-1} = \int_0^x (.)ds$. Applying L^{-1} to equation (3.1.30) gives

$$\int_0^x (\frac{du_i}{dx})ds + L^{-1}N_i(x, u_1, u_2, \cdots, u_n) = L^{-1}g_i, \qquad i = 1, 2, \cdots, n$$
$$u_i(x) - u_i(0) + L^{-1}N_i(x, u_1, u_2, \cdots, u_n) = L^{-1}g_i, \quad i = 1, 2, \cdots, n \tag{3.1.31}$$

The Adomian technique consists of approximating the solution of (3.1.31) by an infinite series

$$u_i = \sum_{j=0}^{\infty} u_{i,j}$$

and the nonlinear terms

$$N_i(x, u_1, u_2, \cdots, u_n) = \sum_{j=0}^{\infty} A_{i,j}(u_{i,0}, u_{i,1}, \cdots, u_{i,j})$$

where $A_{i,j}$ are the Adomian polynomials. Thus (3.1.31) become

$$\sum_{j=0}^{\infty} u_{i,j} = u_i(0) - \sum_{j=0}^{\infty} L^{-1}A_{i,j}(u_{i,0}, u_{i,1}, \cdots, u_{i,j}) + L^{-1}g_i, i = 1, 2, \cdots, n \tag{3.1.32}$$

then we define:

$$u_{i,0} = u_i(0) + L^{-1}g_i$$
$$u_{i,n+1} = -L^{-1}A_{i,n}(u_{i,0}, u_{i,1}, \cdots, u_{i,n})$$
$$n = 0, 1, 2, \cdots \qquad and\ i = 1, 2, \cdots$$

Example 3.1.5. *[12] Consider the following nonlinear system of differential equations*

$$u_1' = 2u_2^2$$
$$u_2' = e^{-x}u_1$$
$$u_3' = u_2 + u_3 \tag{3.1.33}$$

with exact solutions $u_1(x) = e^{2x}$, $u_2(x) = e^x$ *and* $u_3(x) = xe^x$.

Applying $L^{-1} = \int_0^x (.)ds$ *to both sides of* (3.1.33)*:*

$$\begin{aligned} u_1 - u_1(0) &= \int_0^x 2u_2^2 ds \\ u_2 - u_2(0) &= \int_0^x e^{-s} u_1 ds \\ u_3 - u_3(0) &= \int_0^x (u_2 + u_3) ds \end{aligned} \tag{3.1.34}$$

from AMD assume that $u_i = \sum_{n=0}^{\infty} u_{i,n}(x)$ *where* $i = 1, 2, 3$ *and from the first equation of* (3.1.33) *the nonlinear term is*

$N_2(x, u_1, u_2, u_3) = u_2^2 = \sum_{n=0}^{\infty} A_{2,n}$. *Substituting these values in* (3.1.34)

$$\begin{aligned} \sum_{n=0}^{\infty} u_{1,n}(x) &= 1 + 2\sum_{n=0}^{\infty} \int_0^x A_{2,n} ds \\ \sum_{n=0}^{\infty} u_{2,n}(x) &= 1 + \sum_{n=0}^{\infty} \int_0^x e^{-s} u_{1,n} ds \\ \sum_{n=0}^{\infty} u_{3,n}(x) &= \sum_{n=0}^{\infty} \int_0^x (u_{3,n} + u_{2,n}) ds \end{aligned}$$

This leads to the following scheme:

$$\begin{aligned} u_{1,0} &= 1 \quad u_{1,n+1} = 2\int_0^x A_{2,n} ds \\ u_{2,0} &= 1 \quad u_{2,n+1} = \int_0^x e^{-s} u_{1,n} ds \\ u_{3,0} &= 0 \quad u_{3,n+1} = \int_0^x (u_{3,n} + u_{2,n}) ds \end{aligned}$$

where $n = 0, 1, 2, \cdots$. *In order to determine* $A_{2,n}$ *we use the Adomian's formula(2.1.10). Then the first two iterations of approximate solutions of the system above are for* $n = 0$,

$u_{1,1} = 2\int_0^x A_{2,0} ds$

$A_{2,0} = N_2(u_{2,0}) = (u_{2,0})^2 = 1$

then $u_{1,1} = 2\int_0^x 1 ds = 2x$

$u_{2,1} = \int_0^x e^{-s} u_{1,0} ds = \int_0^x e^{-s} ds = -e^{-x} + 1$

$u_{3,1} = \int_0^x (u_{3,0} + u_{2,0}) ds = \int_0^x 1 ds = x$

for $n = 1$,

$u_{1,2} = 2 \int_0^x A_{2,1} ds$

$A_{2,1} = \frac{1}{1!} \frac{d}{d\lambda^1} [N_2(u_{2,0} + \lambda u_{2,1})]_{\lambda=0}$

$= \frac{1}{1!} \frac{d}{d\lambda^1} [(u_{2,0} + \lambda u_{2,1})^2]_{\lambda=0}$

$= \frac{1}{1!} \frac{d}{d\lambda^1} [(u_{2,0}^2 + \lambda^2 u_{2,1}^2 + 2\lambda u_{2,0} u_{2,1})]_{\lambda=0}$

$= \frac{1}{1!} \frac{d}{d\lambda^1} [1 + \lambda^2 (1 - e^{-x}) + 2\lambda (1 - e^{-x})]_{\lambda=0}$

$= [2(1 - e^{-x})(-e^{-x}) + 2\lambda + 2\lambda(-e^{-x}) + 2(1 - \lambda e^{-x})]_{\lambda=0}$

$= -2e^{-x} + 2$

then,

$u_{1,2} = 2 \int_0^x (-2e^{-s} + 2) ds = 4e^{-x} + 4x - 4$ $u_{2,2} = \int_0^x e^{-s} u_{1,1} ds$

$= \int_0^x e^{-s} (2s) ds$

$= -2xe^{-x} - 2e^{-x} + 2$

$u_{3,2} = \int_0^x (u_{3,1} + u_{2,1}) ds$

$= \int_0^x (s + -e^{-s} + 1) ds = \frac{x^2}{2} - e^{-x} + x + 1$

Thus u_1, u_2 *and* u_3 *are approximately given*

$$u_1 = u_{1,0} + u_{1,1} + u_{1,2} = 2x + 4e^{-x} + 4x - 3$$

$$u_2 = u_{2,0} + u_{2,1} + u_{2,2} = -e^{-x} - 2xe^{-x} - 2e^{-x} + 4$$

$$u_3 = u_{3,0} + u_{3,1} + u_{3,2} = x + \frac{x^2}{2} - e^{-x} + x + 1$$

3.2 AMD For Solving Partial Differential Equation

It known that the nonlinear partial differential equations describe very large branches of science and engineering applications. Much research like [7, 29, 8, 40]has been worked to get numerical solutions of these types of problems when they have some computational difficulties and usually roundoff error causes loss of accuracy but the AMD need only few computation.

3.2.1 First Order nonlinear PDE

Refereing to [7], the most general form of nonlinear first order partial differential equation in one dimension

$$F(u, u_x, u_t, x, t) = 0$$

with initial condition:
$u(x, 0) = f(x), \qquad \forall x \in \Omega$
and subject to the boundary condition:
$u(x, t) = \psi(x, t), \qquad \forall x \in \partial\Omega$
where Ω is the region of solution and $\partial\Omega$ is the boundary of Ω.
The following examples illustrate how can we apply the ADM to solve first order PDEs.

Example 3.2.1. *[7] Consider the nonlinear hyperbolic equation*

$$\frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x}, \qquad 0 < x \leq 1 \quad 0 \leq t \leq 1 \tag{3.2.1}$$

with initial condition:
$u(x, 0) = \frac{x}{10}, \qquad 0 < x \leq 1$

The exact solution of this problem is

$$u(x,t) = \frac{x}{10-t}$$

Solution using ADM

Rewrite (3.2.1) *in operator form*

$$L_t u = Nu \tag{3.2.2}$$

where $L_t = \frac{\partial}{\partial t}$ *and its inverse is* $L_t^{-1} = \int_0^t (.) d\tau$ *and* $Nu = u\frac{\partial u}{\partial x}$. *Applying* L_t^{-1} *to both sides of* (3.2.2) *we obtain*

$$\begin{aligned} L_t^{-1} L_t u &= L_t^{-1} Nu \\ \int_0^t (\frac{\partial u}{\partial \tau}) d\tau = u(x,\tau) \mid_0^t &= L_t^{-1} Nu \\ u(x,t) - u(x,0) &= L_t^{-1} Nu \end{aligned}$$

then,

$$u(x,t) = u(x,0) + L_t^{-1} Nu \tag{3.2.3}$$

after that substitute $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$ *and* $Nu = \sum_{n=0}^{\infty} A_n$ *and the initial condition* $u(x,0) = \frac{x}{10}$ *in* (3.2.3)

$$\sum_{n=0}^{\infty} u_n(x,t) = \frac{x}{10} + L_t^{-1} \sum_{n=0}^{\infty} A_n$$

Thus, the recursive scheme is

$$u_0(x,t) = \frac{x}{10}$$

$$u_{n+1}(x,t) = L_t^{-1} A_n, \qquad n = 0,\ 1,\ 2,\ 3, \cdots$$

to determine the Adomian polynomials A_n*'s we use the Adomian formula*(2.1.10).

Where $Nu = u\frac{\partial u}{\partial x}$, *so the first four Adomian polynomials and series terms are*

$$\begin{aligned}
A_0 &= N(u_0) = u_0\frac{\partial u_0}{\partial x} = \frac{x}{10}\frac{1}{10}\\
u_1 &= L_t^{-1}A_0 = \int_0^t (\frac{x}{10^2})d\tau = \frac{xt}{10^2}\\
A_1 &= \frac{1}{1!}\frac{d}{d\lambda}[N(u_0+u_1\lambda)]_{\lambda=0} = \frac{1}{1!}\frac{d}{d\lambda}[(u_0+u_1\lambda)\frac{\partial(u_0+u_1\lambda)}{\partial x}]_{\lambda=0}\\
&= \frac{d}{d\lambda}[(\frac{x}{10}+\frac{xt}{10^2}\lambda)(\frac{1}{10}+\frac{t}{10^2})]_{\lambda=0} = \frac{d}{d\lambda}[\frac{x}{10^2}+\frac{\lambda xt}{10^3}+\frac{\lambda xt}{10^3}+\frac{\lambda^2 xt}{10^4}]_{\lambda=0}\\
&= [\frac{xt}{10^3}+\frac{xt}{10^3}+\frac{2\lambda xt}{10^4}]_{\lambda=0} = \frac{2xt}{10^3}\\
u_2 &= L_t^{-1}A_1 = \int_0^t (\frac{2x\tau}{10^3})d\tau = \frac{xt^2}{10^3}\\
A_2 &= \frac{1}{2!}\frac{d^2}{d\lambda^2}[N(u_0+u_1\lambda+u_2\lambda^2)]_{\lambda=0}\\
&= \frac{1}{2!}\frac{d^2}{d\lambda^2}[(u_0+u_1\lambda+u_2\lambda^2)\frac{\partial(u_0+u_1\lambda+u_2\lambda^2)}{\partial x}]_{\lambda=0}\\
&= \frac{1}{2!}\frac{d^2}{d\lambda^2}[(\frac{x}{10}+\frac{xt}{10^2}\lambda+\frac{xt^2}{10^3}\lambda^2)\frac{\partial(\frac{x}{10}+\frac{xt}{10^2}\lambda+\frac{xt^2}{10^3}\lambda^2)}{\partial x}]_{\lambda=0}\\
&= \frac{3xt^2}{10^4}\\
u_3 &= L_t^{-1}A_2 = \int_0^t (\frac{3x\tau^2}{10^4})d\tau = \frac{xt^3}{10^4}\\
A_3 &= \frac{1}{3!}\frac{d^3}{d\lambda^3}[N(u_0+u_1\lambda+u_2\lambda^2+u_3\lambda^3)]_{\lambda=0}\\
&= \frac{1}{3!}\frac{d^3}{d\lambda^3}[(u_0+u_1\lambda+u_2\lambda^2+u_3\lambda^3)\frac{\partial(u_0+u_1\lambda+u_2\lambda^2+u_3\lambda^3)}{\partial x}]_{\lambda=0}\\
&= \frac{1}{3!}\frac{d^3}{d\lambda^3}[(\frac{x}{10}+\frac{xt}{10^2}\lambda+\frac{xt^2}{10^3}\lambda^2+\frac{xt^3}{10^4}\lambda^3)\frac{\partial(\frac{x}{10}+\frac{xt}{10^2}\lambda+\frac{xt^2}{10^3}\lambda^2+\frac{xt^3}{10^4}\lambda^3)}{\partial x}]_{\lambda=0}\\
&= \frac{4xt^3}{10^5}\\
u_4 &= L_t^{-1}A_3 = \int_0^t (\frac{4x\tau^3}{10^5})d\tau = \frac{xt^4}{10^5}
\end{aligned}$$

Thus

$$\begin{aligned}u(x,t) &= \sum_{n=0}^{\infty} u_n(x,t)\\ &= u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+u_4(x,t)+\cdots\\ &= \frac{x}{10}+\frac{xt}{10^2}+\frac{xt^2}{10^3}+\frac{xt^3}{10^4}+\frac{xt^4}{10^5}+\cdots\\ &= \frac{x}{10}[1+\frac{t}{10}+\frac{t^2}{10^2}+\frac{t^3}{10^3}+\frac{t^4}{10^4}+\cdots]\end{aligned}$$

the series in the brackets above is a geometric series and its sum is $\sum_{n=0}^{\infty}(t/10)^n = \frac{1}{1-\frac{t}{10}}$, *then*

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \frac{x/10}{1-\frac{t}{10}} = \frac{x}{10-t}$$

which is the exact solution.

Example 3.2.2. *[7] Consider this problem*

$$\frac{\partial u}{\partial t} = x^2 - \frac{1}{4}(\frac{\partial u}{\partial x})^2, \qquad 0 < x \le 1 \quad 0 \le t \le 1 \tag{3.2.4}$$

with initial condition:

$u(x,0) = 0, \qquad 0 < x \le 1$

The exact solution of this problem is

$$u(x,t) = x^2 tanh(t)$$

(3.2.4) *in decomposition method operator form is*

$$L_t u = g(x) + Nu \tag{3.2.5}$$

where $L_t = \frac{\partial}{\partial t}$ *and its inverse is* $L_t^{-1} = \int_0^t (.)d\tau$, $Nu = -\frac{1}{4}(\frac{\partial u}{\partial x})^2$*and* $g(x) = x^2$. *Applying* L_t^{-1} *to both sides of* (3.2.5)

$$\begin{aligned}L_t^{-1}L_t u &= L_t^{-1}g(x) + L_t^{-1}Nu\\ \int_0^t (\frac{\partial u}{\partial \tau})d\tau = u(x,\tau)\mid_0^t &= L_t^{-1}g(x) + L_t^{-1}Nu\\ u(x,t) - u(x,0) &= L_t^{-1}g(x) + L_t^{-1}Nu\end{aligned}$$

then,

$$u(x,t) = u(x,0) + L_t^{-1}g(x) + L_t^{-1}Nu$$

Substitute $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$ *and* $Nu = \sum_{n=0}^{\infty} A_n$ *and the initial condition in the equation above, we get*

$$\sum_{n=0}^{\infty} u_n(x,t) = x^2t + L_t^{-1}\sum_{n=0}^{\infty} A_n$$

Thus, the recursive solution terms are

$$u_0(x,t) = x^2t$$

$$u_{n+1}(x,t) = L_t^{-1}A_n, \qquad n = 0,\ 1,\ 2,\cdots \tag{3.2.6}$$

According to Adomian formula(2.1.10) *we find these values of* A_n*'s and then we can* *find u_n's terms from relation* (3.2.6)

$$\begin{aligned}
A_0 &= N(u_0) = -\frac{1}{4}(\frac{\partial u_0}{\partial x})^2 = -x^2t^2 \\
u_1 &= L_t^{-1}A_0 = \int_0^t (-x^2\tau^2)d\tau = \frac{-1}{3}x^2t^3 \\
A_1 &= \frac{1}{1!}\frac{d}{d\lambda}[N(u_0+u_1\lambda)]_{\lambda=0} = \frac{1}{1!}\frac{d}{d\lambda}[(-\frac{1}{4}(\frac{\partial(u_0+u_1\lambda)}{\partial x})^2]_{\lambda=0} \\
&= \frac{d}{d\lambda}[(-\frac{1}{4}(2xt-\frac{2}{3}xt^3\lambda))^2]_{\lambda=0} = \frac{2}{3}x^2t^4 \\
u_2 &= L_t^{-1}A_1 = \int_0^t (\frac{2}{3}x^2\tau^4)d\tau = \frac{2}{15}x^2t^5 \\
A_2 &= \frac{1}{2!}\frac{d^2}{d\lambda^2}[N(u_0+u_1\lambda+u_2\lambda^2)]_{\lambda=0} \\
&= \frac{-1}{9}x^2t^6 + \frac{-4}{15}x^2t^6 = \frac{-51}{135}x^2t^6 \\
u_3 &= L_t^{-1}A_2 = \int_0^t (\frac{-51}{135}x^2\tau^6)d\tau = \frac{-51}{945}x^2t^7 \\
&= \frac{-17}{315}x^2t^7 \\
A_3 &= \frac{1}{3!}\frac{d^3}{d\lambda^3}[N(u_0+u_1\lambda+u_2\lambda^2+u_3\lambda^3)]_{\lambda=0} \\
&= \frac{4}{45}x^2t^8 + \frac{34}{315}x^2t^8 = \frac{62}{315}x^2t^8 \\
u_4 &= L_t^{-1}A_3 = \int_0^t (\frac{62}{315}x^2\tau^8)d\tau = \frac{62}{2835}x^2t^9
\end{aligned}$$

Thus

$$\begin{aligned}
u(x,t) &= \sum_{n=0}^{\infty} u_n(x,t) \\
&= u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + u_4(x,t) + \cdots \\
&= x^2t + \frac{-1}{3}x^2t^3 + \frac{2}{15}x^2t^5 + \frac{-17}{315}x^2t^7 + \frac{62}{2835}x^2t^9 + \cdots \\
&= x^2[t + \frac{-1}{3}t^3 + \frac{2}{15}t^5 + \frac{-17}{315}t^7 + \frac{62}{2835}t^9 + \cdots] = x^2\tanh(t)
\end{aligned}$$

which is the exact solution.

3.2.2 Second Order PDE's

In this section we consider the ADM to solve linear and nonlinear heat[14, 27] and wave equations[15].

Consider the **linear** heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + q(x,t) \tag{3.2.7}$$

where $0 < x < 1$ and $0 < t \leq T$

with initial condition

$u(x,0) = f(x), \qquad 0 < x < 1$

and with nonlocal boundary conditions

$u(0,t) = \int_0^1 \phi(x,t)u(x,t)dx + g_1(t)$

$u(1,t) = \int_0^1 \psi(x,t)u(x,t)dx + g_2(t)$, $0 < t \leq T$

where $f(x)$, $g_1(t)$, $g_2(t)$, $\phi(x,t)$and $\psi(x,t)$ are given smooth functions, and T is constant.

Using the ADM technique to solve this kind of problems gives results with high accuracy and much closer to the exact solution or it gives the exact solution.

Rewrite (3.2.7) in ADM operator form we get

$$L_t u = L_{xx} u + q(x,t) \tag{3.2.8}$$

where $L_t = \frac{\partial}{\partial t}$ and its inverse is $L_t^{-1} = \int_0^t (.)d\tau$ and $L_{xx} = \frac{\partial^2}{\partial x^2}$. Take L_t^{-1} to both sides of (3.2.8)

$$L_t^{-1} L_t u = L_t^{-1}(L_{xx} u + q(x,t))$$

$$\int_0^t (\frac{\partial u}{\partial \tau})d\tau = u(x,\tau)\mid_0^t = L_t^{-1}(L_{xx} u + q(x,t))$$

$$u(x,t) - u(x,0) = L_t^{-1}(L_{xx} u + q(x,t))$$

then,

$$u(x,t) = f(x) + L_t^{-1}(L_{xx}u + q(x,t)).$$

Substituting the decomposition series $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$ we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) + L_t^{-1}(L_{xx} \sum_{n=0}^{\infty} u_n(x,t) + q(x,t))$$

Thus, the algorithm of the solution is

$$\begin{aligned} u_0(x,t) &= f(x) + L_t^{-1} q(x,t) \\ u_{n+1}(x,t) &= L_t^{-1} L_{xx} u_n, \ n = 0, \ 1, \ 2, \cdots \end{aligned} \tag{3.2.9}$$

Example 3.2.3. *[14] Consider the following equation:*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{-2(x^2+t+1)}{(t+1)^3}$$

where $0 < x < 1$ *and* $0 < t \leq 1$

with initial condition

$u(x,0) = x^2$

and with nonlocal boundary conditions

$u(0,t) = \int_0^1 xu(x,t)dx + \frac{1}{4(t+1)^2}$

$u(1,t) = \int_0^1 xu(x,t)dx + \frac{3}{4(t+1)^2}, \ 0 < t \leq 1$

the exact solution is $u(x,t) = \frac{x^2}{(t+1)^2}$.

We use (3.2.9) *to get the solution iterations*

$$\begin{aligned} u_0(x,t) &= x^2 + L_t^{-1} \frac{-2(x^2+t+1)}{(t+1)^3} \\ &= x^2 + \int_0^t \frac{-2(x^2+\tau+1)}{(\tau+1)^3} d\tau \\ &= x^2 + \int_0^t \frac{-2x^2}{(\tau+1)^3} d\tau + \int_0^t \frac{-2(\tau+1)}{(\tau+1)^3} d\tau \\ &= x^2 + \frac{x^2}{(\tau+1)^2} \ |_0^t + \frac{2}{(\tau+1)} \ |_0^t \\ &= \frac{x^2}{(t+1)^2} + \frac{2}{(t+1)} - 2 \end{aligned}$$

and

$u_{n+1}(x,t) = L_t^{-1} L_{xx} u_n, \ n =, \ 1, \ 2, \cdots$

for $n = 1,$

$u_1 = L_t^{-1} L_{xx} u_0 = L_t^{-1}(\frac{2}{(t+1)^2} = \int_0^t (\frac{2}{(\tau+1)^2}) d\tau = \frac{-2}{(t+1)} + 2$

and

$u_2 = L_t^{-1} L_{xx} u_1 = L_t^{-1}(0) = 0$

Thus $u_n = 0$ *for all* $n = 2, \ 3, \cdots.$

The final result is

$$u(x,t) = u_0 + u_1 = \frac{x^2}{(t+1)^2} + \frac{2}{(t+1)} - 2 + \frac{-2}{(t+1)} + 2 = \frac{x^2}{(t+1)^2}$$

which is the exact solution of this problem.

Example 1. *(Linear Heat equation with Dirichlet boundary conditions[33])*
Consider the following linear heat equation subject to Dirichlet BC's:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - hu = 0 \qquad 0 < x < \pi, \quad t > 0,$$
$$u(0,t) = 0, \quad u(\pi,t) = 1 \qquad t \geq 0,$$
$$u(x,0) = 0, \qquad 0 < x < \pi \tag{3.2.10}$$

where h *is a constant,* $h > 0$*.*

Solution: *The ADM operator form of* (3.2.10) *is given by*

$$L_t u = L_{xx} u + Ru \tag{3.2.11}$$

where $L_t = \frac{\partial}{\partial t}$*,* $L_{xx} = \frac{\partial^2}{\partial x^2}$ *and the remainder linear term* $Ru = hu$*. Applying* $L_{xx}^{-1} = \int_0^x \int_0^x (.) ds ds$ *on both sides of* (3.2.11)

$$L_{xx}^{-1} L_{xx} u = L_{xx}^{-1} L_t u + L_{xx}^{-1} Ru$$
$$\int_0^x \int_0^x (u_{xx}) ds ds = L_{xx}^{-1} L_t u + L_{xx}^{-1} Ru$$
$$\int_0^x (u_x(s,t) - u_x(0,t)) ds ds = L_{xx}^{-1} L_t u + L_{xx}^{-1} Ru$$
$$u(s,t) \mid_0^x - u_x(0,t) s \mid_0^x = L_{xx}^{-1} L_t u + L_{xx}^{-1} Ru$$

then,

$$u(x,t) - u(0,t) - u_x(0,t)x = L_{xx}^{-1} L_t u + L_{xx}^{-1} Ru. \tag{3.2.12}$$

From the boundary condition $u(0,t) = 0$*, but* $u_x(0,t)$ *is not given we can use the boundary condition* $u(\pi,t) = 1$ *to get it. Using* (3.2.12) *at* $x = \pi$ *we get that*
$u(\pi,t) - u_x(0,t)\pi = L_{xx}^{-1} L_t u(\pi,t) + L_{xx}^{-1} R(\pi,t)$
$u_x(0,t)\pi = u(\pi,t) - \int_0^\pi \int_0^\pi L_t u(\pi,t) dsds - \int_0^\pi \int_0^\pi hu(\pi,t) = 1 - 0 - h\frac{\pi^2}{2}$*, then we have*

$$u_x(0,t) = \frac{1}{\pi} - h\frac{\pi}{2},$$

substituting these values in (3.2.12) *we have*

$$u(x,t) = [\frac{1}{\pi} - h\frac{\pi}{2}]x + L_{xx}^{-1} L_t u + L_{xx}^{-1} Ru.$$

According to the ADM, the solution $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$*. Thus* (3.2.12) *in series form is*

$$\sum_{n=0}^{\infty} u_n(x,t) = [\frac{1}{\pi} - h\frac{\pi}{2}]x + L_{xx}^{-1} L_t \sum_{n=0}^{\infty} u_n(x,t) + L_{xx}^{-1} R \sum_{n=0}^{\infty} u_n(x,t).$$

Thus, the algorithm of the solution is

$$u_0(x,t) = [\frac{1}{\pi} - h\frac{\pi}{2}]x,$$
$$u_{n+1}(x,t) = L_{xx}^{-1}(L_t u_n + Ru_n) \quad n = 0,\ 1,\ 2, \cdots .$$

For simplification, replace the constant $[\frac{1}{\pi} - h\frac{\pi}{2}]$ *with k, thus the first three iterations* *are*

$$
\begin{aligned}
u_1 &= L_{xx}^{-1}(L_t u_0 + Ru_0) = L_{xx}^{-1}(0 + hkx) = \int_0^x \int_0^x (hks)dsds \\
&= hk\frac{s^2}{2} \mid_0^x = hk\frac{x^2}{2}. \\
u_2 &= L_{xx}^{-1}(L_t u_1 + Ru_1) = L_{xx}^{-1}(0 + k(h)^2\frac{x^2}{2}) = \int_0^x \int_0^x (k(h)^2\frac{s^2}{2})dsds \\
&= k(h)^2\frac{s^4}{4!} \mid_0^x = k(h)^2\frac{x^4}{4!}. \\
u_3 &= L_{xx}^{-1}(L_t u_0 + Ru_1) = L_{xx}^{-1}(0 + k(h)^3\frac{x^4}{4!}) = \int_0^x \int_0^x (k(h)^3\frac{s^4}{4!})dsds \\
&= k(h)^3\frac{s^6}{6!} \mid_0^x = k(h)^3\frac{x^6}{6!}.
\end{aligned}
$$

we can conclude that the n^{th} term is given by

$$u_n = k(h)^n \frac{x^{2n}}{(2n)!}.$$

Thus,

$$
\begin{aligned}
u(x,t) &= u_0 + u_1 + u_2 + u_3 + \cdots + u_n + \cdots \\
&= kx + hk\frac{x^2}{2} + k(h)^2\frac{x^4}{4!} + k(h)^3\frac{x^6}{6!} + \cdots + k(h)^n\frac{x^{2n}}{(2n)!} + \cdots \\
&= k\sum_{n=0}^{\infty}(h)^n\frac{x^{2n}}{(2n)!} \\
&= [\frac{1}{\pi} - h\frac{\pi}{2}]\sum_{n=0}^{\infty}(h)^n\frac{x^{2n}}{(2n)!}.
\end{aligned}
$$

Example 2. *(Nonlinear heat equation[27]) Consider the* ***nonlinear*** *heat equation described by*

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= (A(u)u_x)_x + C(u) \qquad (3.2.13) \\
u(x,0) &= g(x)
\end{aligned}
$$

where $A(u)$ and $C(u)$ are arbitrary given functions.

In the operator form, equation(3.2.13) can be written as

$$L_t u = N(u) \qquad (3.2.14)$$

where $L_t = \frac{\partial}{\partial t}$ *and its inverse is* $L_t^{-1} = \int_0^t (.)d\tau$ *and* $N(u) = (A(u)u_x)_x + C(u)$. *Take* L_t^{-1} *to both sides of* (3.2.14)

$$
\begin{aligned}
L_t^{-1} L_t u &= L_t^{-1}(N(u)) \\
\int_0^t (\frac{\partial u}{\partial \tau}) d\tau = u(x,\tau) \mid_0^t &= L_t^{-1}(N(u)) \\
u(x,t) - u(x,0) &= L_t^{-1}(N(u))
\end{aligned}
$$

then,

$$u(x,t) = g(x) + L_t^{-1}(N(u))$$

Substituting $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$, $Nu = \sum_{n=0}^{\infty} A_n$ *and the initial condition in equation above to obtain*

$$\sum_{n=0}^{\infty} u_n(x,t) = g(x) + L_t^{-1} \sum_{n=0}^{\infty} A_n$$

Thus, the recursive scheme is given by

$$
\begin{aligned}
u_0(x,t) &= g(x) \\
u_{n+1}(x,t) &= L_t^{-1} A_n, \quad n = 0,\ 1,\ 2, \cdots
\end{aligned}
$$

Wave Equation

In this section we will study the solution of linear and nonlinear wave equations subject to initial conditions and with well defined boundary conditions using the ADM,[29].

Example 3. *(Linear wave equation[29]) Consider the following linear wave equation*

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2} && 0 < x < \pi,\ t > 0,\\
u(0,t) &= u(\pi,t) = 0 && t > 0,\\
u(x,0) &= \sin^3(x) && 0 < x < \pi\\
u_t(x,0) &= \sin(2x) && 0 < x < \pi. \quad (3.2.15)
\end{aligned}$$

Solution:

Rewrite (3.2.15) *in ADM operator form*

$$L_{tt}u = L_{xx}u, \tag{3.2.16}$$

where $L_{tt} = \frac{\partial^2}{\partial t^2}$ *and* $L_{xx} = \frac{\partial^2}{\partial x^2}$. *Operating each side of* (3.2.16) *by* $L_{tt}^{-1} = \int_0^t \int_0^t (.) d\tau d\tau$

$$\begin{aligned}
L_{tt}^{-1} L_{tt} u &= L_{tt}^{-1} L_{xx} u\\
\int_0^t \int_0^t \left(\frac{\partial^2 u(x,\tau)}{\partial t^2}\right) d\tau d\tau &= L_{tt}^{-1} L_{xx} u\\
\int_0^t \left(\frac{\partial u(x,\tau)}{\partial t} \mid_0^\tau\right) d\tau &= L_{tt}^{-1} L_{xx} u\\
\int_0^t \left(\frac{\partial u(x,\tau)}{\partial t} - \frac{\partial u(x,0)}{\partial t}\right) d\tau &= L_{tt}^{-1} L_{xx} u\\
u(x,\tau) \mid_0^\tau - u_t(x,0)\tau \mid_0^\tau &= L_{tt}^{-1} L_{xx} u
\end{aligned}$$

then,

$$u(x,t) - u(x,0) - u_t(x,0)t = L_{tt}^{-1} L_{xx} u.$$

Substituting the initial conditions we have

$$u(x,t) = \sin^3(x) + t\sin(2x) + L_{tt}^{-1}L_{xx}u.$$

Decompose the solution into infinite series

$$\sum_{n=0}^{\infty} u_n(x,t) = \sin^3(x) + t\sin(2x) + L_{tt}^{-1}L_{xx}\sum_{n=0}^{\infty} u_n(x,t).$$

Then we get this algorithm of solution

$$\begin{aligned} u_0(x,t) &= \sin^3(x) + t\sin(2x) \\ u_n(x,t) &= L_{tt}^{-1}L_{xx}u_n(x,t) \qquad n = 0,\ 1,\ 2,\ 3,\ldots. \end{aligned}$$

For the previous equation, the first two iterations are

$$\begin{aligned} u_1 &= L_{tt}^{-1}L_{xx}u_0(x,t) = L_{tt}^{-1}\frac{\partial}{\partial x}[3\sin^2(x)\cos(x) + 2t\cos(2x)] \\ &= \int_0^t\int_0^t [6\sin(x)\cos^2(x) - 3\sin^3(x) - 4\tau\sin(2x)]d\tau d\tau \\ &= 3t^2\sin(x)\cos^2(x) - 3\frac{t^2}{2}\sin^3(x) - 2\frac{t^3}{3}\sin(2x) \\ u_2 &= L_{tt}^{-1}L_{xx}u_1(x,t) \\ &= L_{tt}^{-1}[-6t^2\sin^2(x)\cos(x) + 3t^2\cos^3(x) - \frac{9}{2}t^2\sin^2(x)\cos(x) - \frac{4}{3}t^3\cos(2x)]d\tau d\tau \\ &= -\frac{t^4}{2}\sin^2(x)\cos(x) + \frac{t^4}{4}\cos^3(x) - \frac{3}{8}t^4\sin^2(x)\cos(x) - \frac{1}{15}t^5\cos(2x). \end{aligned}$$

Thus,

$$\begin{aligned} u(x,t) &= u_0 + u_1 + u_2 + \cdots \\ &= \sin^3(x) + t\sin(2x) + 3t^2\sin(x)\cos^2(x) - 3\frac{t^2}{2}\sin^3(x) - 2\frac{t^3}{3}\sin(2x) - \frac{t^4}{2}\sin^2(x)\cos(x) \\ &+ \frac{t^4}{4}\cos^3(x) - \frac{3}{8}t^4\sin^2(x)\cos(x) - \frac{1}{15}t^5\cos(2x) + \cdots \end{aligned}$$

Example 3.2.4. *(Nonlinear wave equation[15]) Consider the nonhomogeneous nonlinear partial differential equation*

$$\frac{\partial^2 u}{\partial x^2} - u\frac{\partial^2 u}{\partial t^2} = \phi(x,t) \tag{3.2.17}$$

with initial condition

$$u(0,t) = f(t), \quad u_x(0,t) = g(t)$$

Rewrite (3.2.17) *in the operator form:*

$$\begin{aligned} L_{xx}u - N(u) &= \phi(x,t) \\ L_{xx}u &= N(u) + \phi(x,t) \end{aligned} \tag{3.2.18}$$

where $L_{xx} = \frac{\partial^2}{\partial x^2}$*, then its inverse is* $L_{xx}^{-1} = \int_0^x \int_0^x (.)dsds$ *and* $N(u) = uu_{tt}$*. Applying* L_{xx}^{-1} *to* (3.2.18) *yields*

$$\begin{aligned} L_{xx}^{-1}L_{xx}u &= L_{xx}^{-1}N(u) + L_{xx}^{-1}\phi(x,t) \\ &= \int_0^x \int_0^x (\frac{\partial^2 u}{\partial s^2})dsds \\ &= \int_0^x (\frac{\partial u}{\partial s} - c_1)ds \end{aligned}$$

thus,

$$u(x,t) - c_1x - c_2 = L_{xx}^{-1}N(u) + L_{xx}^{-1}\phi(x,t) \tag{3.2.19}$$

using the initial conditions

$$c_2 = f(t) \quad \textit{and} \quad c_1 = g(t)$$

substituting in (3.2.19)

$$u(x,t) = g(t)x + f(t) + L_{xx}^{-1}N(u) + L_{xx}^{-1}\phi(x,t)$$

using that $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$ *and* $Nu = \sum_{n=0}^{\infty} A_n$

$$\sum_{n=0}^{\infty} u_n(x,t) = g(t)x + f(t) + L_{xx}^{-1}\sum_{n=0}^{\infty} A_n + L_{xx}^{-1}\phi(x,t)$$

then it follows that

$$\begin{aligned} u_0(x,t) &= g(t)x + f(t) + L_{xx}^{-1}\phi(x,t) \\ u_{n+1}(x,t) &= L_{xx}^{-1}A_n, n = 0,\ 1,\ 2, \cdots \end{aligned}$$

Numerical Example: Consider the following nonhomogeneous nonlinear wave problem

$$\frac{\partial^2 u}{\partial x^2} - u\frac{\partial^2 u}{\partial t^2} = 2 - 2(t^2 + x^2)$$

with initial condition

$u(x, 0) = x^2$

$u(0, t) = t^2, \quad u_x(0, t) = 0$

In this problem $Nu = u\frac{\partial^2 u}{\partial t^2}$, $\phi(x, t) = 2 - 2(t^2 + x^2)$, $f(t) = t^2$ and $g(t) = 0$. Using the same procedure and the recursive terms algorithm obtained in the previous description we get

$$\begin{aligned} u_0(x,t) &= g(t)x + f(t) + L_{xx}^{-1}\phi(x,t) = t^2 + L_{xx}^{-1}(2 - 2(x^2 + t^2) \\ &= t^2 + \int_0^x \int_0^x (2 - 2(s^2 + t^2)dsds = t^2 + x^2 - \frac{x^4}{6} - x^2t^2 \end{aligned}$$

and

$$\begin{aligned} u_1 &= L_{xx}^{-1}A_0 = L_{xx}^{-1}N(u_0) = L_{xx}^{-1}u_0\frac{\partial^2 u_0}{\partial t^2} \\ &= \int_0^x \int_0^x (t^2 + s^2 - \frac{s^4}{6} - s^2t^2)\frac{\partial^2(t^2 + s^2 - \frac{s^4}{6} - s^2t^2)}{\partial t^2}dsds \\ &= \int_0^x \int_0^x (t^2 + s^2 - \frac{s^4}{6} - s^2t^2)(2 - 2s^2)dsds \\ &= x^2t^2 + \frac{1}{6}x^4 - \frac{1}{3}x^4t^2 - \frac{7}{90}x^6 + \frac{2}{15}x^6t^2 + \frac{x^8}{16} \end{aligned}$$

The second term is

$$\begin{aligned}
u_2 &= L_{xx}^{-1} A_1 \\
A_1 &= \frac{d}{d\lambda}[N(u_0 + u_1\lambda)] \mid_{\lambda=0} \\
&= \frac{d}{d\lambda}[(u_0 + u_1\lambda)\frac{\partial^2(u_0 + u_1\lambda)}{\partial t^2}] \mid_{\lambda=0} \\
&= u_1\frac{\partial^2(u_0}{\partial t^2} + u_0\frac{\partial^2(u_1)}{\partial t^2} \\
&= 2x^2t^2 + \frac{1}{3}x^4 - \frac{2}{3}x^4t^2 - \frac{7}{45}x^6 + \frac{2}{15}x^6t^2 + \frac{1}{84}x^8 \\
&- 2x^4t^2 - \frac{1}{3}x^6 + \frac{2}{3}x^6t^2 + \frac{7}{45}x^8 - \frac{2}{15}x^8t^2 - \frac{1}{10}x^{10} \\
&+ \frac{4}{3}x^4 + \frac{2}{15}x^8 + \frac{4}{3}x^2t^2 + \frac{2}{15}x^6t^2 - \frac{4}{3}x^4t^2 \\
&- \frac{2}{15}x^8t^2 - \frac{2}{9}x^6 - \frac{1}{45}x^{10}
\end{aligned}$$

so

$u_2 = L_{xx}^{-1} A_1 = \frac{1}{3}x^4t^2 + \frac{7}{90}x^6 - \frac{2}{15}x^6t^2 - \frac{x^8}{16} + \cdots$

It can be easily observed that the self canceling "noise" terms appear between the recursive components. So the solution is

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = x^2 + t^2$$

3.2.3 Third Order nonlinear Partial Differential Equations

In this section I apply the ADM to solve the generalized log-KdV equation as an example of higher order partial differential equations after reduced it to log-KdV equation from [38, 13].

generalized log-KdV equation

The Korteweg de Vries equation (KdV) is a mathematical model of waves on shallow water surfaces. It is an example of non-linear partial differential equation whose solutions can be exactly and precisely specified. The mathematical theory behind the KdV equation is a topic of active research. The KdV equation was first introduced by Boussinesq in 1877 and rediscovered by Diederik Korteweg and Gustav de Vries in 1895.

The log-KdV equation models solitary waves in anharmonic chains with Hertzian interaction forces and it is defined by

$$v_t + (v \ln \mid v \mid)_x + v_{xxx} = 0. \tag{3.2.20}$$

In this section we will try to solve the generalized log-KdV equation by using the ADM. The generalized log-KdV equation is given by

$$v_t + (v \ln \mid v \mid^n)_x + v_{xxx} = 0, \qquad n = 1,\ 2, \cdots . \tag{3.2.21}$$

Solution

Using the rescaling of space x and time t variables as

$$x \to \sqrt{n}x,$$
$$t \to \sqrt{n^3}t,$$

carries out the equation(3.2.21) to equation(3.2.20).

In order to solve (3.2.20), firstly we will use the transform

$$u(x,t) = \ln v(x,t),$$

this is the same as

$$v(x,t) = e^{u(x,t)}, \tag{3.2.22}$$

and then substituting it in (3.2.20) to get

$$u_t e^u + (e^u u)_x + (e^u)_{xxx} = 0,$$

after differentiating the second and the third terms we get

$$[u_t + u_{xxx} + uu_x + u_x + 3u_x u_{xx} + (u_x)^3]e^u = 0, \tag{3.2.23}$$

since e^u cant equal zero, so (3.2.23) holds if

$$u_t + u_{xxx} + uu_x + u_x + 3u_x u_{xx} + (u_x)^3 = 0. \tag{3.2.24}$$

subject to the initial condition

$$u(x,0) = \frac{c}{k} + \frac{1}{2} - \frac{x^2}{4}$$

where c and k are nonzero constants.

Rewrite (3.2.24) in the Adomian operator form

$$\begin{aligned} L_t u + L_{xxx} u + L_x u + N(u) &= 0 \\ L_t u &= -L_{xxx} u - L_x u - N(u) \end{aligned} \tag{3.2.25}$$

where $L_t = \frac{\partial}{\partial t}$, $L_{xxx} = \frac{\partial^3}{\partial x^3}$, $L_x = \frac{\partial}{\partial x}$ and
$N(u) = uu_x + 3u_x u_{xx} + (u_x)^3$. Applying $L_t^{-1} = \int_0^t (.) d\tau$ to both sides of (3.2.25) to obtain

$$\begin{aligned} L^{-1} L_t u = \int_0^t u_t d\tau &= u(x,y,t) - u(x,y,0) \\ &= L_t^{-1}[-L_{xxx} u - L_x u - N(u)] \end{aligned}$$

Using the decomposition series for u and the Adomian polynomial representation for the nonlinear term $N(u)$, gives

$$\sum_{n=0}^{\infty} u_n(x,t) = u(x,0) + L_t^{-1}[-L_{xxx} \sum_{n=0}^{\infty} u_n(x,t) - L_x \sum_{n=0}^{\infty} u_n(x,t) - \sum_{n=0}^{\infty} A_n]$$

where A_n's are Adomian polynomials. Then the ADM iterative scheme is

$$\begin{aligned} u_0 &= u(x,0) = \frac{c}{k} + \frac{1}{2} - \frac{x^2}{4}, \\ u_{n+1} &= L_t^{-1}[-L_{xxx} u_n(x,t) - L_x u_n(x,t) - A_n]. \end{aligned}$$

By considering $N(u) = uu_x + 3u_x u_{xx} + (u_x)^3$, the first three iterations are

$$u_1 = L_t^{-1}[-L_{xxx}u_0(x,t) - L_x u_0(x,t) - A_0] = \int_0^t (0 + \frac{x}{2} - A_0)d\tau$$

$$A_0 = N(u_0) = u_0(u_0)_x + 3(u_0)_x(u_0)_{xx} + ((u_0)_x)^3$$
$$= (\frac{c}{k} + \frac{1}{2} - \frac{x^2}{4})(-\frac{x}{2}) + 3(-\frac{x}{2})\frac{-1}{2} + (-\frac{x}{2})^3$$
$$= -\frac{cx}{2k} - \frac{x}{4} + \frac{x^3}{8} + \frac{3x}{4} - \frac{x^3}{8} = -\frac{cx}{2k} + \frac{x}{2}$$

then,$u_1 = \int_0^t (\frac{x}{2} - \frac{cx}{2k} - \frac{x}{2})d\tau = \frac{cxt}{2k}$

$$u_2 = L_t^{-1}[-L_{xxx}u_1(x,t) - L_x u_1(x,t) - A_1] = \int_0^t (0 + \frac{ct}{2k} - A_0)d\tau$$

$$A_1 = u_1 N'(u_0) = u_1 \frac{\partial}{\partial x}(u_0(u_0)_x + 3(u_0)_x(u_0)_{xx} + ((u_0)_x)^3)$$
$$= u_1[u_0(u_0)_{xx} + ((u_0)_x)^2 + 3(u_0)_x(u_0)_{xxx} + 3((u_0)_{xx})^2 + 3((u_0)_x)^2(u_0)_{xx}]$$
$$= \frac{cxt}{2k}[(\frac{c}{k} + \frac{1}{2} - \frac{x^2}{4})\frac{-1}{2} + \frac{x^2}{4} + 0 + 3(\frac{x^2}{4})(\frac{-1}{2})]$$
$$= -\frac{c^2xt}{(2k)^2} - \frac{cxt}{8k}$$

then, $u_2 = \int_0^t (\frac{c\tau}{2k} + \frac{c^2x\tau}{(2k)^2} + \frac{cx\tau}{8k})d\tau$
$= \frac{ct^2}{4k} + \frac{c^2xt^2}{2(2k)^2} + \frac{cxt^2}{16k}$.

$$u_3 = L_t^{-1}[-L_{xxx}u_2(x,t) - L_x u_2(x,t) - A_2]$$
$$= \int_0^t (0 + \frac{c^2t + ckt^2}{16k^2} - A_0)d\tau$$
$$A_2 = u_2 N'(u_0) + \frac{1}{2!}u_1 N''(u_0)$$
$$= (\frac{ct^2}{4k} + \frac{c^2xt^2}{2(2k)^2} + \frac{cxt^2}{16k})(\frac{-c}{k} + \frac{1}{2}) + \frac{1}{2!}(\frac{cxt}{2k})(0)$$
$$= \frac{-c^2t^2}{4k^2} + \frac{-c^3xt^2}{8k^3} + \frac{-c^2xt^2}{16k^2} + \frac{ct^2}{8k} + \frac{c^2xt^2}{16k^2} + \frac{cxt^2}{32k}$$

then, $u_3 = \int_0^t (\frac{-c^2\tau^2}{4k^2} + \frac{-c^3x\tau^2}{8k^3} + \frac{-c^2x\tau^2}{16k^2} + \frac{c\tau^2}{8k} + \frac{c^2x\tau^2}{16k^2} + \frac{cx\tau^2}{32k})d\tau$
$= \frac{-c^2t^3}{12k^2} + \frac{-c^3xt^3}{24k^3} + \frac{-c^2xt^3}{48k^2} + \frac{ct^3}{24k} + \frac{c^2xt^3}{48k^2} + \frac{cxt^3}{96k}$.

The solution in the series form is thus given by

$$\begin{aligned}
u(x,t) &= u_0 + u_1 + u_2 + u_3 + \cdots \\
&= \frac{c}{k} + \frac{1}{2} - \frac{x^2}{4} + \frac{cxt}{2k} + \frac{ct^2}{4k} + \frac{c^2xt^2}{2(2k)^2} \\
&+ \frac{cxt^2}{16k} + \frac{-c^2t^3}{12k^2} + \frac{-c^3xt^3}{24k^3} + \frac{-c^2xt^3}{48k^2} \\
&+ \frac{ct^3}{24k} + \frac{c^2xt^3}{48k^2} + \frac{cxt^3}{96k} + \cdots .
\end{aligned}$$

To find the solution $v(x,t)$ we use (3.2.22).

3.2.4 System of Partial Differential Equations

In this section we apply the ADM to solve system of partial differential equations[7]. Consider the following system of partial differential equations

$$\begin{aligned}
u_t &= uu_x + vu_y \\
v_t &= vv_x + uv_y
\end{aligned} \tag{3.2.26}$$

with the initial conditions

$$u(x,y,0) = v(x,y,0) = x + y$$

The exact solution is

$$u(x,y,t) = v(x,y,t) = \frac{x+y}{(1-2t)}$$

write the system in (3.2.26) in Adomian operator form

$$\begin{aligned}
L_t u &= N(u) + K(u) \\
L_t v &= K(v) + N(v)
\end{aligned} \tag{3.2.27}$$

where $L_t = \frac{\partial}{\partial t}$, $N(u) = uu_x$, $K(u) = vu_y$, $N(v) = vv_y$ and $K(v) = uv_x$. Apply $L^{-1}(t) = \int_0^t (.)d\tau$ to both sides of (3.2.27) to obtain

$$L^{-1}L_t u = \int_0^t u_t d\tau = u(x,y,t) - u(x,y,0) = L^{-1}(t)(N(u) + K(u))$$
$$L^{-1}L_t v = \int_0^t v_t d\tau = v - v(x,y,0) = L^{-1}(t)(K(v) + N(v)) \quad (3.2.28)$$

By Adomian decomposition method the solution of the above system assume to be at the series form

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n(x,y,t)$$
$$v(x,y,t) = \sum_{n=0}^{\infty} v_n(x,y,t) \quad (3.2.29)$$

and the nonlinear terms are

$$N(u) = uu_x = \Sigma_{n=0}^{\infty} A_n$$
$$N(v) = vv_x = \Sigma_{n=0}^{\infty} B_n$$
$$K(u) = vu_y = \Sigma_{n=0}^{\infty} C_n$$
$$K(v) = uv_y = \Sigma_{n=0}^{\infty} D_n \quad (3.2.30)$$

the values of A_n's, B_n's, C_n's and D_n's are determined using the general formula of Adomian polynomials(2.1.10).

Substitute (3.2.29) and (3.2.30) in (3.2.28) we get

$$\Sigma_{n=0}^{\infty} u_n(x,y,t) - u(x,y,0) = L^{-1}(\Sigma_{n=0}^{\infty} A_n + \Sigma_{n=0}^{\infty} C_n)$$
$$\Sigma_{n=0}^{\infty} v_n(x,y,t) - v(x,y,0) = L^{-1}(\Sigma_{n=0}^{\infty} B_n + \Sigma_{n=0}^{\infty} D_n)$$

the solution is obtained by the following scheme

$$u_0 = x + y, \qquad u_{n+1} = L^{-1}(A_n + C_n)$$
$$v_0 = x + y, \qquad v_{n+1} = L^{-1}(B_n + D_n)$$

The first four terms are

$$A_0 = N(u_0) = u_0(u_0)_x = x + y$$

$$C_0 = K(u_0) = v_0(u_0)_y = x + y$$

so,$u_1 = 2(x + y)t$

$$B_0 = N(v_0) = v_0(v_0)_x = x + y$$

$$D_0 = K(v_0) = u_0(v_0)_y = x + y$$

so,$v_1 = 2(x + y)t$

$$A_1 = u_1(u_0)_x + u_0(u_1)_x = 4(x + y)t$$

$$C_1 = v_0(u_1)_y + v_1(u_0)_y = 4(x + y)t$$

so,$u_2 = (x + y)(2t)^2$

$$B_1 = v_1(v_0)_x + v_0(v_1)_x = 4(x + y)t$$

$$D_1 = u_0(v_1)_y + u_1(v_0)_y = 4(x + y)t$$

so,$v_2 = (x + y)(2t)^2$

$$A_2 = u_0(u_2)_x + u_1(u_1)_x + u_2(u_0)_x = 12(x + y)t^2$$

$$C_2 = v_0(u_2)_y + v_1(u_1)_y + v_2(u_0)_y = 12(x + y)t^2$$

so,$u_3 = (x + y)(2t)^3$

$$B_2 = v_0(v_2)_x + v_1(v_1)_x + v_2(v_0)_x = 12(x + y)t^2$$

$$D_2 = u_0(v_2)_y + u_1(v_1)_y + u_2(v_0)_y = 12(x + y)t^2$$

so,$v_3 = (x + y)(2t)^3$

$$A_3 = u_0(u_3)_x + u_1(u_2)_x + u_2(u_1)_x + u_3(u_0)_x = 4(x + y)(2t)^2$$

$$C_3 = v_0(u_3)_y + v_1(u_2)_y + v_2(u_1)_y + v_3(u_0)_y = 4(x + y)(2t)^2$$

so,$u_4 = (x+y)(2t)^4$

$$B_3 = v_0(v_3)_x + v_1(v_2)_x + v_2(v_1)_x + v_3(v_0)_x = 4(x+y)(2t)^2$$

$$D_3 = u_0(v_3)_y + u_1(v_2)_y + u_2(v_1)_y + u_3(v_0)_y = 4(x+y)(2t)^2$$

so,$v_4 = (x+y)(2t)^4$

$$\vdots$$

$$u_n = (x+y)(2t)^n$$

$$v_n = (x+y)(2t)^n$$

$$\vdots$$

Thus the solution is

$$u(x,y,t) = v(x,y,t) = \Sigma_{n=0}^{\infty}(x+y)(2t)^n = \frac{x+y}{(1-2t)}$$

which is the exact solution of the system in (3.2.26).

3.3 Integral Equations

Stating from the 1980s, the ADM has been applied to a wide class of integral equations[2, 17] To illustrate the procedure, consider the following Volterra integral equations of the second kind given by

$$u(x) = f(x) + \lambda \int_a^x K(s,t)[L(u(s)) + N(u(s))]ds, \qquad \lambda \neq 0 \tag{3.3.1}$$

Where $f(x)$ is a given function, λ is a parameter, $K(x,t)$ is the kernel, $L(u(x))$ and $N(u(x))$ are linear and nonlinear operators respectively. Assume that the solution

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

and

$$N(u(x)) = \sum_{n=0}^{\infty} A_n$$

substituting these assumptions in (3.3.1)

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_a^x K(s,t)[L(\sum_{n=0}^{\infty} u_n(s)) + \sum_{n=0}^{\infty} A_n]ds$$

this gives the following scheme

$$u_0 = f(x)$$
$$u_{n+1} = \lambda \int_a^x K(s,t)[L(u_n(s)) + A_n]ds, \quad n = 0,\ 1,\ 2, \cdots$$

Example 3.3.1. *[2] Consider the nonlinear Volterra integral equation*

$$u(x) = x + \int_0^x u^2(s)ds.$$

Matching this equation with the general integral equation form we have

$$f(x) = x, \qquad\qquad N(u(x)) = u^2(x).$$

According to the techniques described above, we have the following recursive relationship

$$u_0 = x,$$
$$u_{n+1} = \int_0^x A_n(s)ds, \quad n = 0,\ 1,\ 2,\ 3, \cdots .$$

We obtain the first three iterations as

$$u_1 = \int_0^x A_0(s)ds$$
$$A_0 = N(u_0(x)) = u_0^2(x) = x^2$$

so, $u_1 = \int_0^x s^2 ds = \frac{x^3}{3}$

$$u_2 = \int_0^x A_1(s)ds$$
$$A_1 = u_1 N'(u_0(x)) = \frac{x^3}{3}(2x) = \frac{2x^4}{3}$$

$so, u_2 = \int_0^x \frac{2s^4}{3} ds = \frac{2x^5}{15}$

$$
\begin{aligned}
u_3 &= \int_0^x A_2(s)ds \\
A_2 &= u_2 N'(u_0(x) + \frac{1}{2!} u_1 N''(u_0(x)) \\
&= \frac{2x^5}{15}(2x) + \frac{1}{2!}\frac{x^3}{3}(2)\frac{51x^6}{135}
\end{aligned}
$$

$so, u_3 = \int_0^x \frac{51x^6}{135} ds = \frac{17x^5}{315}$ *Thus,*

$u(x) = u_0 + u_1 + u_2 + u_3 + \cdots = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^5}{315} + \cdots = \tan(x)$

CHAPTER 4

Inverse Parabolic Problems

In this chapter ADM is applied for solving some inverse problems in some inverse parabolic problems. In partial differential equations problems, solving an equation with initial conditions and boundary conditions that are all specified completely, these problems are called direct problems and these problems are well posed. In other words in these problems we have one output for each input given. However, when initial or/and boundary conditions or some input coefficient or source function are not given or not completely specified so that the problem has more than one unknown, this problem is called inverse problem, that is given certain output to get unknown input. We call inverse problem of coefficient identification if the problem coefficient is unknown and we call an inverse problem of source identification if the source function is unknown, and so on. Also inverse problems are classified according to the type of the partial equations we have. That means if the partial differential equation is parabolic, then we have parabolic inverse problem, and so on. In this chapter we focus our study on Parabolic Inverse Problems[**?**, 31, 35, 18].

4.1 Inverse Problem of Boundary Conditions Identification

D.Lesnic and L.Elliott [31] apply the ADM to find the temperature and the heat flux at the boundary $x = 0$ from the boundary conditions at $x = 1$.
Consider a **one-dimensional** inverse problem heat conduction

$$u_t = u_{xx} \qquad 0 < x < 1, \quad t > 0 \tag{4.1.1}$$

with the temperature and the heat flux $f_0(t)$ and $g_0(t)$ respectively on the boundary $x = 0$ are unknown, and the temperature and the heat flux $f_1(t)$ and $g_1(t)$ at the boundary $x = 1$ are measured and so they are known

$$u(1,t) = f_1(t), \qquad \frac{\partial u(1,t)}{\partial x} = g_1(t), \qquad t > 0$$

in this case we have overspecification at one boundary which is $x = 1$.
In order to solve this problem the decomposition method is used. Take the inverse operator L_{xx}^{-1} as follow

$$L_{xx}^{-1} = \int_1^x \int_1^x (.) ds ds$$

Applying L_{xx}^{-1} to both sides of the Adomian operator form of (4.1.1) to get

$$L_{xx}^{-1} L_{xx} u = L_{xx}^{-1} L_t u$$

$$\int_1^x \int_1^x (u_{xx}) ds ds = L_{xx}^{-1} L_t u$$

$$\int_1^x (u_x(s,t) - u_x(1,t)) ds = L_{xx}^{-1} L_t u$$

$$u(x,t) - c_1(x-1) - c_2 = L_{xx}^{-1} L_t u \tag{4.1.2}$$

from the boundary conditions at $x = 1$ we get that

$$u(1,t) = c_2 = f_1(t)$$

and

$$\frac{\partial u(1,t)}{\partial x} = g_1(t) = c_1,$$

so (4.1.2) becomes

$$u(x,t) = g_1(t)(x-1) + f_1(t) + L_{xx}^{-1} L_t u.$$

Using $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$ we have

$$\sum_{n=0}^{\infty} u_n(x,t) = g_1(t)(x-1) + f_1(t) + L_{xx}^{-1} L_t \sum_{n=0}^{\infty} u_n(x,t). \tag{4.1.3}$$

From (4.1.3) we can define

$$u_0 = g_1(t)(x-1) + f_1(t).u_{n+1} = L_{xx}^{-1} L_t u_n(x,t), \quad n = 0,\ 1,\ 2, \ldots. \tag{4.1.4}$$

Based on equation(4.1.4), we can calculate

$$\begin{aligned}
u_1 &= L_{xx}^{-1} L_t u_0(x,t) = L_{xx}^{-1} L_t (g_1(t)(x-1) + f_1(t)) \\
&= \int_1^x \int_1^x (g_1'(t)s + f_1'(t)) ds ds \\
&= \frac{(x-1)^3}{3!} g_1'(t) + \frac{(x-1)^2}{2!} f_1'(t) \\
u_2 &= L_{xx}^{-1} L_t u_1(x,t) = L_{xx}^{-1} L_t (\frac{(x-1)^3}{3!} g_1'(t) + \frac{(x-1)^2}{2!} f_1'(t)) \\
&= \int_1^x \int_1^x (g_1''(t)\frac{s^3}{3!} + f_1''(t)\frac{s^2}{2!}) ds ds \\
&= \frac{(x-1)^5}{5!} g_1''(t) + \frac{(x-1)^4}{4!} f_1''(t) \\
u_3 &= L_{xx}^{-1} L_t u_2(x,t) = L_{xx}^{-1} L_t (\frac{(x-1)^5}{5!} g_1''(t) + \frac{(x-1)^4}{4!} f_1''(t)) \\
&= \int_1^x \int_1^x g_1'''(t)\frac{s^5}{5!} + f_1'''(t)\frac{s^4}{4!}) ds ds \\
&= \frac{(x-1)^7}{7!} g_1'''(t) + \frac{(x-1)^6}{6!} f_1'''(t)
\end{aligned}$$

$$\vdots$$

$$u_n = \frac{(x-1)^{n+1}}{(n+1)!} g_1^{(n)}(t) + \frac{(x-1)^n}{n!} f_1^{(n)}(t)$$

Thus,

$$
\begin{aligned}
u &= u_0 + u_1 + u_2 + u_3 + \cdots + u_n + \cdots \\
&= g_1(t)(x-1) + f_1(t) + \frac{(x-1)^3}{3!} g_1'(t) + \frac{(x-1)^2}{2!} f_1'(t) \\
&+ \frac{(x-1)^5}{5!} g_1''(t) + \frac{(x-1)^4}{4!} f_1''(t) \\
&+ \frac{(x-1)^7}{7!} g_1'''(t) + \frac{(x-1)^6}{6!} f_1'''(t) + \cdots \\
&+ \frac{(x-1)^{n+1}}{(n+1)!} g_1^{(n)}(t) + \frac{(x-1)^n}{n!} f_1^{(n)}(t) \\
&= [g_1(t)(x-1) + \frac{(x-1)^3}{3!} g_1'(t) + \frac{(x-1)^5}{5!} g_1''(t) \\
&+ \frac{(x-1)^7}{7!} g_1'''(t) + \cdots + \frac{(x-1)^{n+1}}{(n+1)!} g_1^{(n)}(t) + \cdots]
\end{aligned}
$$

$$
\begin{aligned}
&+ [f_1(t) + \frac{(x-1)^2}{2!} f_1'(t) + \frac{(x-1)^4}{4!} f_1''(t) \\
&+ \frac{(x-1)^6}{6!} f_1'''(t) + \cdots + \frac{(x-1)^n}{n!} f_1^{(n)}(t) + \cdots].
\end{aligned}
$$

The general formula is

$$
u(x,t) = \sum_{n=0}^{\infty} \frac{(x-1)^{2n+1}}{(2n+1)!} g_1^{(n)}(t) + \sum_{n=0}^{\infty} \frac{(x-1)^{2n}}{(2n)!} f_1^{(n)}(t) \tag{4.1.5}
$$

To check that this result is the solution of (4.1.1) and satisfies the boundary conditions we differentiate it twice with respect to x and once with respect to t we get

$$\begin{aligned} u_x(x,t) &= \sum_{n=1}^{\infty}(2n+1)\frac{(x-1)^{2n}}{(2n+1)!}g_1^{(n)}(t) + \sum_{n=1}^{\infty}(2n)\frac{(x-1)^{2n-1}}{(2n)!}f_1^{(n)}(t) \\ &= \sum_{n=1}^{\infty}\frac{(x-1)^{2n}}{(2n)!}g_1^{(n)}(t) + \sum_{n=1}^{\infty}\frac{(x-1)^{2n-1}}{(2n-1)!}f_1^{(n)}(t) \\ u_{xx}(x,t) &= \sum_{n=2}^{\infty}(2n)\frac{(x-1)^{2n-1}}{(2n)!}g_1^{(n)}(t) + \sum_{n=2}^{\infty}(2n-1)\frac{(x-1)^{2n-2}}{(2n-1)!}f_1^{(n)}(t) \\ &= \sum_{n=2}^{\infty}\frac{(x-1)^{2n-1}}{(2n-1)!}g_1^{(n)}(t) + \sum_{n=2}^{\infty}\frac{(x-1)^{2n-2}}{(2n-2)!}f_1^{(n)}(t) \\ u_t(x,t) &= \sum_{n=1}^{\infty}\frac{(x-1)^{2n+1}}{(2n+1)!}g_1^{(n+1)}(t) + \sum_{n=1}^{\infty}\frac{(x-1)^{2n}}{(2n)!}f_1^{(n+1)}(t) \end{aligned}$$

let $m = n+1$ then $n = m-1$ if $n = 1 \to m = 2$ so

$$\begin{aligned} u_t(x,t) &= \sum_{n=2}^{\infty}\frac{(x-1)^{2(n-1)+1}}{(2(n-1)+1)!}g_1^{(n)}(t) + \sum_{n=1}^{\infty}\frac{(x-1)^{2(n-1)}}{(2(n-1))!}f_1^{(n)}(t) \\ &= \sum_{n=2}^{\infty}\frac{(x-1)^{2n-1}}{(2n-1)!}g_1^{(n)}(t) + \sum_{n=2}^{\infty}\frac{(x-1)^{2n-2}}{(2n-2)!}f_1^{(n)}(t) = u_{xx} \end{aligned}$$

the boundary conditions are satisfied, for $n = 0$ and $x = 1$

$$u(x,t) = f_1(t) + (x-1)g_1(t)$$

so $u(1,t) = f_1(t)$ and $\frac{\partial u(1,t)}{\partial x} = g_1(t)$

Therefor, the result at (4.1.5) is the solution of the inverse problem in heat conduction (4.1.1).

Two dimension inverse problem

Let us now describe the solution of the inverse problem in two dimensions x and y. Consider the generalized 2-D heat conduction equation:

$$u_{xx} + u_{yy} = u_t,\ 0 < x < 1,\ 0 < y < 1,\ t > 0$$

subject to the following boundary conditions

$$u(1,y,t) = f_1(y,t) \qquad \frac{\partial u(1,y,t)}{\partial x} = g_1(y,t)$$

$$u(x,1,t) = f_2(x,t) \qquad \frac{\partial u(x,1,t)}{\partial x} = g_2(x,t)$$

In this case, we have two solutions, one in $x-dim$ and one in $y-dim$ and Adomian shows that the exact solution is the summation of these two solutions divided by 2. In $x-dim$ we take $L_{xx}^{-1} = \int_1^x \int_1^x (.)dsds$ to both sides of equation above

$$\begin{aligned}
L_{xx}^{-1} L_{xx} u &= L_{xx}^{-1}[L_t u - L_{yy} u] \\
\int_1^x \int_1^x (u_{xx}) dsds &= L_{xx}^{-1}[L_t u - L_{yy} u] \\
\int_1^x (u_x(s) - u_x(1)) dsds &= L_{xx}^{-1}[L_t u - L_{yy} u] \\
u(x,y,t) - c_1(x-1) - c_2 &= L_{xx}^{-1}[L_t u - L_{yy} u] \\
u(x,y,t) - \phi_x &= L_{xx}^{-1}[L_t u - L_{yy} u],
\end{aligned}$$

where $\phi_x = c_1(x-1) + c_2$ and from the boundary conditions at $x = 1$ we find that $c_1 = g_1(y,t)$ and $c_2 = f_1(y,t)$, then $\phi_x = g_1(y,t)(x-1) + f_1(y,t)$, so

$$u(x,y,t) = \phi_x + L_{xx}^{-1}[L_t u - L_{yy} u].$$

Decompose the solution u in infinite series with $u_0 = \phi_x$ to obtain the following recursive relationship of the solution:

$$u_{n+1}(x,y,t) = [L_{xx}^{-1} L_t - L_{xx}^{-1} L_{yy}] u_n.$$

The same calculations are made for the $y-dim$. Define $L_{yy}^{-1} = \int_1^y \int_1^y (.)dsds$

$$\begin{aligned}
L_{yy}^{-1} L_{yy} u &= L_{yy}^{-1}[L_t u - L_{xx} u] \\
\int_1^y \int_1^y (u_{yy}) dsds &= L_{yy}^{-1}[L_t u - L_{xx} u] \\
\int_1^y (u_y(s) - u_y(1)) dsds &= L_{yy}^{-1}[L_t u - L_{xx} u] \\
u(x,y,t) - c_3(y-1) - c_4 &= L_{yy}^{-1}[L_t u - L_{xx} u] \\
u(x,y,t) - \phi_y &= L_{yy}^{-1}[L_t u - L_{xx} u],
\end{aligned}$$

where $\phi_y = c_3(y-1)+c_4$, from the boundary conditions at $y=1$ we get $c_3 = g_2(x,t)$ and $c_4 = f_2(x,t)$, then

$$u(x,y,t) = g_2(x,t)(y-1) + f_2(x,t) + L_{yy}^{-1}[L_t u - L_{xx}u].$$

Using $u_0 = \phi_y$, we get

$$u_{n+1}(x,y,t) = [L_{yy}^{-1}L_t - L_{yy}^{-1}L_{xx}]u_n.$$

The recursive relationship of the exact solution is

$$\begin{aligned} u_0 &= \frac{1}{2}[\phi_x + \phi_y] \\ u_{n+1}(x,y,t) &= \frac{1}{2}[L_{yy}^{-1}L_t + L_{xx}^{-1}L_t - L_{yy}^{-1}L_{xx} - L_{xx}^{-1}L_{yy}]u_n \end{aligned}$$

Example 4.1.1. *Consider a two-dimensional inverse problem*

$$u_{xx} + u_{yy} = u_t,\ 0 < x < 1,\ 0 < y < 1,\ t > 0$$

subject to the following boundary conditions

$$u(1,y,t) = y^2 + 4t + 1 \qquad \frac{\partial u(1,y,t)}{\partial x} = 2$$

$$u(x,1,t) = x^2 + 4t + 1 \qquad \frac{\partial u(x,1,t)}{\partial x} = 2$$

Using the above solution, the first four iterations are:

$$\begin{aligned} u_0 &= \frac{1}{2}[\phi_x + \phi_y] = \frac{1}{2}[g_1(y,t)(x-1) + f_1(y,t) + g_2(x,t) \\ &(y-1) + f_2(x,t)] \\ &= \frac{1}{2}[y^2 + 4t + 1 + 2(x-1) + x^2 + 4t + 1 + 2(y-1)] \\ &= x + y + 4t + \frac{x^2+y^2}{2} - 1 \end{aligned}$$

$$
\begin{aligned}
u_1 &= \frac{1}{2}[L_{yy}^{-1}L_t + L_{xx}^{-1}L_t - L_{yy}^{-1}L_{xx} - L_{xx}^{-1}L_{yy}]u_0 \\
&= \frac{1}{2}[L_{yy}^{-1}(4) + L_{xx}^{-1}(4) - L_{yy}^{-1}(1) - L_{xx}^{-1}(1)] \\
&= \frac{1}{2}(\frac{3(x-1)^2}{2} + \frac{3(y-1)^2}{2}) = \frac{3(x-1)^2}{4} + \frac{3(y-1)^2}{4} \\
&= \frac{3(x^2+y^2)}{4} - \frac{3(y+x)}{2} + \frac{3}{2} \\
u_2 &= \frac{1}{2}[L_{yy}^{-1}L_t + L_{xx}^{-1}L_t - L_{yy}^{-1}L_{xx} - L_{xx}^{-1}L_{yy}]u_1 \\
&= \frac{1}{2}[L_{yy}^{-1}(0) + L_{xx}^{-1}(0) - L_{yy}^{-1}(\frac{3}{2}) - L_{xx}^{-1}(\frac{3}{2})] \\
&= \frac{1}{2}(-(\frac{3}{2})\frac{(x-1)^2}{2} - (\frac{3}{2})\frac{(y-1)^2}{2}) \\
&= \frac{-3(x^2+y^2)}{8} + \frac{3(y+x)}{4} - \frac{3}{4} \\
u_3 &= \frac{1}{2}[L_{yy}^{-1}L_t + L_{xx}^{-1}L_t - L_{yy}^{-1}L_{xx} - L_{xx}^{-1}L_{yy}]u_2 \\
&= \frac{1}{2}[L_{yy}^{-1}(0) + L_{xx}^{-1}(0) - L_{yy}^{-1}(\frac{-3}{4}) - L_{xx}^{-1}(\frac{-3}{4})] \\
&= \frac{1}{2}((\frac{3}{4})\frac{(x-1)^2}{2} + (\frac{3}{4})\frac{(y-1)^2}{2}) \\
&= \frac{3(x^2+y^2)}{16} - \frac{3(y+x)}{8} + \frac{3}{8} \\
&\vdots \\
u_n &= \frac{3(-1)^{n+1}}{2^{n+1}}(x^2+y^2) + \frac{3(-1)^n}{2^n}(y+x) + \frac{3(-1)^n}{2^n}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
u &= u_0 + u_1 + u_2 + u_3 + \cdots + u_n + \cdots \\
&= u_0 + \sum_{n=1}^{\infty} \frac{3(-1)^{n+1}}{2^{n+1}}(x^2+y^2) + \sum_{n=1}^{\infty} \frac{3(-1)^n}{2^n}(y+x) + \sum_{n=1}^{\infty} \frac{3(-1)^n}{2^n} \\
&= 4t + x + y + \frac{x^2+y^2}{2} - 1 + \sum_{n=1}^{\infty} \frac{3(-1)^{n+1}}{2^{n+1}}(x^2+y^2) \\
&+ \sum_{n=1}^{\infty} \frac{3(-1)^n}{2^n}(y+x) + \sum_{n=1}^{\infty} \frac{3(-1)^n}{2^n} \\
&= 4t + (x^2+y^2)[\frac{1}{2} + \frac{3}{4}\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}] + (x+y-1)[1 - \frac{3}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}].
\end{aligned}
$$

since the above are geometric series so the summation is

$$
\begin{aligned}
u(x,y,t) &= 4t + (x^2+y^1)[\frac{1}{2} + \frac{3}{4}\frac{1}{1+(1/2)}] + (y+x-1)[1 - \frac{3}{2}\frac{1}{1+(1/2)}] \\
&= 4t + x^2 + y^2
\end{aligned}
$$

The Inverse Heat Conduction Problem With Mixed Boundary Conditions

Consider the following inverse problem

$$u_t = u_{xx}, \ 0 < x < 1, \ t > 0 \tag{4.1.6}$$

Subject to initial condition

$u(x,0) = p(x)$

and with boundary conditions

$u(0,t) = f_0(t)$ unknown condition

$-\frac{\partial u(0,t)}{\partial x} = g_0(t)$ unknown condition

$u(x_0,t) = h(t)$ known condition

$\frac{\partial u(1,t)}{\partial x} = g_1(t)$ known condition

Where $x_0 \in (0,1)$ if $x_0 = 0$ then we have a direct problem, if $x_0 = 1$ we return to the case discussed in the beginning. We solve this problem in two different approaches. The first approach[31]:

Rewrite (4.1.6) using Adomian operator form

$$L_t u = L_{xx} u$$

Apply the inverse operator $L_t^{-1} = \int_0^t (.)ds$ to the equation above $L_t^{-1} L_t u = L_t^{-1} L_{xx} u$ which implies $u(x,t) - u(x,0) = L_t^{-1} L_{xx} u$

using initial condition $u(x,0) = p(x)$ we have the equation

$$u(x,t) = p(x) + L_t^{-1} L_{xx} u$$

Now, define the inverse operator with respect to x as

$$L_{xx}^{-1} = \int_{x_0}^{x} \int_{1}^{x} (.)dsds.$$

So

$$L_{xx}^{-1} L_{xx} u = L_{xx}^{-1} L_t u$$

then, $u(x,t) - u(x_0,t) - u_x(1,t)(x - x_0) = L_{xx}^{-1} L_t u$ from the boundary conditions at $x = 1$ and $x = x_0$ we get $u(x_0,t) = h(t)$ and $u_x(1,t) = g_1(t)$, then

$$u(x,t) = g_1(x,t)(x - x_0) + h(t) + L_{xx}^{-1} L_t u$$

Then use the following recursive relation

$$u_0 = \frac{1}{2}[p(x) + g_1(x,t)(x - x_0) + h(t)]$$
$$u_{n+1}(x,t) = \frac{1}{2}[L_{xx}^{-1} L_t + L_t^{-1} L_{xx}]u_n$$

Example 4.1.2. *Consider the inverse heat conduction described in* (4.1.6) *with* $h(t) = 2t + x_0^2$, $p(x) = x^2$ *and* $g_1(x,t) = 2$. *Then applying the recursive relation*

(4.1.4) *we have*

$$
\begin{aligned}
u_0 &= \frac{1}{2}[x^2 + 2(x - x_0) + 2t + x_0^2] \\
u_1 &= \frac{1}{2}[L_{xx}^{-1}L_t + L_t^{-1}L_{xx}]u_0 = \frac{1}{4}[L_{xx}^{-1}(2) + L_t^{-1}(2)] \\
&= \frac{1}{4}[x^2 - x_0^2 - 2(x - x_0) + 2t] \\
u_2 &= \frac{1}{2}[L_{xx}^{-1}L_t + L_t^{-1}L_{xx}]u_1 = \frac{1}{8}[L_{xx}^{-1}(2) + L_t^{-1}(2)] \\
&= \frac{1}{8}[x^2 - x_0^2 - 2(x - x_0) + 2t] \\
&\vdots \\
u_n(x,t) &= \frac{1}{2^{n+1}}[x^2 - x_0^2 - 2(x - x_0) + 2t], \ n = 1, \ 2, \ 3, \ldots.
\end{aligned}
$$

Hence using the decomposition series of u *we find*

$$
\begin{aligned}
u &= u_0 + u_1 + u_2 + \cdots + u_n + \cdots \\
&= u_0 + \sum_{n=1}^{\infty} \frac{1}{2^{n+1}}[x^2 - x_0^2 - 2(x - x_0) + 2t] \\
&= \frac{1}{2}[x^2 + 2(x - x_0) + 2t + x_0^2] + \sum_{n=1}^{\infty} \frac{1}{2^{n+1}}[x^2 - x_0^2 - 2(x - x_0) + 2t] \\
&= (x^2 + 2t)\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} + (x - x_0 + \frac{x_0}{2}) + (x_0 - x - \frac{x_0^2}{2})\sum_{n=1}^{\infty} \frac{1}{2^n} \\
&= x^2 + 2t.
\end{aligned}
$$

The Second approach[35]:

In this way we separate our original problem into two problems one is direct in the interval $(x_0, 1)$ and the other is inverse problem in the interval $(0, x_0)$, with $t > 0$. Consider the partial differential equation:

$$u_t = u_{xx}, \quad x_0 < x < 1, \quad t > 0 \tag{4.1.7}$$

subject to the initial condition

$$u(x, 0) = p(x)$$

and boundary conditions

$$u(x_0,t)=h(t),\quad \frac{\partial u(1,t)}{\partial x}=g_1(t).$$

This is the direct heat equation, take $L_t^{-1}=\int_0^t(.)d\tau$ and apply it to (4.1.7) we get that $u(x,t)=p(x)+L_t^{-1}L_{xx}u$ and by substituting $u=\sum_{n=0}^{\infty}u_n$ to arrive the recursive relation:

$$u_0(x,t)=p(x)$$
$$u_{n+1}(x,t)=\int_0^t(L_{xx}u_n)d\tau. \tag{4.1.8}$$

Now consider the following inverse problem

$$u_t=u_{xx},\quad x_0<x<1,\quad t>0$$

subject to the initial condition

$$u(x,0)=p(x)$$

and boundary conditions

$$u(x_0,t)=h(t),\quad \frac{\partial u(0,t)}{\partial x}=g_0(t).$$

Since $g_0(t)$ and also $u(0,t)=f_0(t)$ are unknown. If we integrate (4.1.7) once with respect to x we find that
$\int_0^{x_0}u_{xx}ds=\int_0^{x_0}u_tdx$
$u_x(x_0,t)-u_x(0,t)=\int_0^{x_0}u_tdx$
thus, $u_x(0,t)$ is given by

$$u_x(0,t)=u_x(x_0,t)-\int_0^{x_0}u_tdx. \tag{4.1.9}$$

If we integrate the (4.1.7) twice with respect to x we have
$\int_0^{x_0}\int_0^{x}u_{xx}ds=\int_0^{x_0}\int_0^{x}u_tdx$

$u(x_0,t) - u(0,t) - u_x(0,t)x_0 = \int_0^{x_0}\int_0^{x} u_t dx$
therefor, $u(0,t)$ is obtained from

$$u(0,t) = u(x_0,t) - u_x(0,t)x_0 - \int_0^{x_0} u_t dx. \tag{4.1.10}$$

After computing the approximate value of $u(x,t)$ from the relation (4.1.8) that obtained from the direct part will use for solving $u_x(0,t)$and$u(0,t)$ which they are given by (4.1.9) and (4.1.10), respectively, and then solve the inverse problem.

Example 4.1.3. *[31] For the same example in the first approach we get for the direct problem part*

$$u_0(x,t) = x^2$$
$$u_1(x,t) = \int_0^t (L_{xx}u_0)d\tau = \int_0^t 2dt = 2t$$
$$u_2(x,t) = \int_0^t (L_{xx}u_1)d\tau = 0 \Rightarrow u_n = 0 \quad \forall n = 0,\ 1,\ 2,\ldots$$

thus $u = u_0 + u_1 = x^2 + 2t.$
From (4.1.9) *we get that*
$u_x(0,t) = u_x(x_0,t) - \int_0^{x_0} u_t dx = 2x_0 - \int_0^{x_0} 2dx = 0$
and from (4.1.10)
$u(0,t) = u(x_0,t) - u_x(0,t)x_0 - \int_0^{x_0} u_t dx = 2t + x_0^2 - 0 - \int_0^{x_0} 2dx$
$= 2t + x_0^2 - x_0^2 = 2t.$
We see that results are the same by both approaches.

Inverse Heat Conduction With Dirichlet Conditions

Consider the following Dirichlet inverse problem[31]

$$u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0 \tag{4.1.11}$$

Subject to initial condition
$u(x,0) = p(x)$

and with boundary conditions

$$u(x_0,t) = h(t), \qquad u(1,t) = f_1(t).$$

Where $x_0 \in (0,1)$ if $x_0 = 1$ then the problem has a nonunique solution, while if $x_0 = 0$ have a direct Dirichlet problem for the heat equation.

For solving the Dirichlet problem above using the Adomian decomposition method we first define the inverse operator with respect to x as follow

$$L_{xx}^{-1} = \int_{x_0}^{x}\int_{x_0}^{x}(.)dsds - \frac{x-x_0}{1-x_0}\int_{x_0}^{1}\int_{x_0}^{x}(.)dsds$$

Apply this operator to Adomian operator form of (4.1.11) we get

$$L_{xx}^{-1}L_{xx}u = L_{xx}^{-1}L_t u$$

$$u(x,t) - u(x_0,t) - u_x(x_0,t)(x-x_0) - \frac{x-x_0}{1-x_0}(u(1,t) - u(x_0,t) - u_x(x_0,t)(1-x_0)) = L_{xx}^{-1}L_t u$$

$$u(x,t) - [1 - \frac{x-x_0}{1-x_0}]u(x_0,t) - \frac{x-x_0}{1-x_0}u(1,t) = L_{xx}^{-1}L_t u$$

so the equation of u in $x - dim$ is given by

$$u(x,t) = [1 - \frac{x-x_0}{1-x_0}]u(x_0,t) + \frac{x-x_0}{1-x_0}u(1,t) + L_{xx}^{-1}L_t u.$$

And we know the equation of u with respect to time inverse operator is given by

$$u(x,t) = p(x) + L_t^{-1}L_{xx}u.$$

Taking averages the solution u is

$$u(x,t) = \frac{1}{2}[(1 - \frac{x-x_0}{1-x_0})u(x_0,t) + \frac{x-x_0}{1-x_0}u(1,t) + p(x)] + \frac{1}{2}[L_{xx}^{-1}L_t u + L_t^{-1}L_{xx}u].$$

Substitute the decomposition series $u = \sum_{n=0}^{\infty} u_n$, then the following are the recursive terms of the solution

$$u_0 = \frac{1}{2}[(1 - \frac{x-x_0}{1-x_0})u(x_0,t) + \frac{x-x_0}{1-x_0}u(1,t) + p(x)]$$

$$u_{n+1} = \frac{1}{2}[L_{xx}^{-1}L_t u_n + L_t^{-1}L_{xx}u_n], \quad n = 0,\ 1,\ 2, \ldots.$$

Example 4.1.4. *For the same example in the previous section with* $f_1(t) = 2t + 1$ *we have*

$$\begin{aligned}
u_0 &= \frac{1}{2}[(1 - \frac{x - x_0}{1 - x_0})u(x_0, t) + \frac{x - x_0}{1 - x_0}u(1, t) + p(x)] \\
&= \frac{1}{2}[(1 - \frac{x - x_0}{1 - x_0})(2t + x_0^2) + \frac{x - x_0}{1 - x_0}(2t + 1) + x^2] \\
&= \frac{1}{2}[2t + x_0^2 - x_0^2\frac{x - x_0}{1 - x_0} + \frac{x - x_0}{1 - x_0} + x^2] \\
&= \frac{1}{2}[2t + x_0^2 + (1 - x_0^2)\frac{x - x_0}{1 - x_0} + x^2] \\
&= \frac{1}{2}[2t + x_0^2 + x^2 + (1 + x_0)(x - x_0)] \\
&= \frac{1}{2}[2t + x^2 + x(1 + x_0) - x_0] \\
u_1 &= \frac{1}{2}[L_{xx}^{-1}L_t + L_t^{-1}L_{xx}]u_0 = \frac{1}{4}[L_{xx}^{-1}(2) + L_t^{-1}(2)] \\
&= \frac{1}{4}[x^2 - x_0^2 - 2xx_0 + 2x_0^2 - (x - x_0)(1 + x_0) + 2x_0(x - x_0) + 2t] \\
&= \frac{1}{4}[x^2 - x(1 + x_0) + x_0 + 2t] \\
u_2 &= \frac{1}{2}[L_{xx}^{-1}L_t + L_t^{-1}L_{xx}]u_1 = \frac{1}{8}[x^2 - x(1 + x_0) + x_0 + 2t] \\
&\vdots \\
u_n(x, t) &= \frac{1}{2^{n+1}}[x^2 - x(1 + x_0) + x_0 + 2t], \quad n = 1,\ 2,\ 3, \ldots.
\end{aligned}$$

Thus the final result is

$$\begin{aligned}
u &= u_0 + u_1 + u_2 + \cdots + u_n + \cdots \\
&= \frac{1}{2}[2t + x^2 + x(1 + x_0) - x_0] + \sum_{n=1}^{\infty} \frac{1}{2^{n+1}}[x^2 - x(1 + x_0) + x_0 + 2t] \\
&= (x^2 + 2t)\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} + (x(1 + x_0) - x_0)[1 - \sum_{n=0}^{\infty} \frac{1}{2^{n+1}}] \\
&= x^2 + 2t.
\end{aligned}$$

Inverse Heat Conduction With Neumann Conditions

Consider the following Neumann inverse problem[31]

$$u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0 \tag{4.1.12}$$

Subject to initial condition

$u(x, 0) = p(x)$

and with boundary conditions

$$u_x(x_0, t) = q(t), \qquad u_x(1, t) = g_1(t).$$

Where $x_0 \in (0, 1)$ if $x_0 = 1$ then the problem has a nonunique solution, while if $x_0 = 0$ have a direct Neumann problem for the heat equation. To solve this problem we use the same procedure as in previous section, define L_{xx}^{-1} by

$$L_{xx}^{-1} = \int_{x_0}^{x}\int_{x_0}^{x}(.)dsds - \frac{(x-x_0)^2}{2(1-x_0)}\int_{x_0}^{1}(.)dsds.$$

Apply this operator to Adomian operator form of (4.1.12) get

$L_{xx}^{-1}L_{xx}u = L_{xx}^{-1}L_t u$

$u(x, t) - u(x_0, t) - u_x(x_0, t)(x - x_0) - \frac{(x-x_0)^2}{2(1-x_0)}(u_x(1, t) - u_x(x_0, t) = L_{xx}^{-1}L_t u$

since $u(x_0, t)$ is not given call it $C(t)$, then

$$u(x, t) = C(t) + u_x(x_0, t)(x - x_0) + \frac{(x - x_0)^2}{2(1 - x_0)}(u_x(1, t) - u_x(x_0, t) + L_{xx}^{-1}L_t u$$

Then the exact equation of u after considering the solution with respect to time is

$$\begin{aligned} u(x, t) = &\frac{1}{2}[C(t) + u_x(x_0, t)(x - x_0) + \frac{(x - x_0)^2}{2(1 - x_0)}(u_x(1, t) - u_x(x_0, t) + p(x)] \\ &+ \frac{1}{2}[L_{xx}^{-1}L_t u + L_t^{-1}L_{xx}u]. \end{aligned}$$

Substitute $u = \sum_{n=0}^{\infty} u_n$, then we obtain

$$\begin{aligned} u_0 &= \frac{1}{2}[C(t) + u_x(x_0, t)(x - x_0) + \frac{(x - x_0)^2}{2(1 - x_0)}(u_x(1, t) - u_x(x_0, t) + p(x)] \\ u_{n+1} &= \frac{1}{2}[L_{xx}^{-1}L_t u_n + L_t^{-1}L_{xx}u_n], \quad n = 0,\ 1,\ 2, \ldots. \end{aligned}$$

Example 4.1.5. *For the same example in the previous section we have*

$$\begin{aligned}
u_0 &= \frac{1}{2}[C(t) + u_x(x_0,t)(x-x_0) + \frac{(x-x_0)^2}{2(1-x_0)}(u_x(1,t) - u_x(x_0,t) + p(x)] \\
&= \frac{1}{2}[C(t) + 2x_0(x-x_0) + \frac{(x-x_0)^2}{2(1-x_0)}(2-2x_0) + x^2] \\
&= \frac{1}{2}[C(t) + 2x^2 - x_0^2] \\
u_1 &= \frac{1}{2}[L_{xx}^{-1}L_t + L_t^{-1}L_{xx}]u_0 \\
&= \frac{1}{4}[L_{xx}^{-1}(C'(t)) + L_t^{-1}(2)] = t \\
u_2 &= 0 \\
u_n(x,t) &= 0, \quad \forall n = 2,\ 3,\ 4,\dots
\end{aligned}$$

Thus the final result is

$$u = u_0 + u_1 = \frac{1}{2}[C(t) + 2x^2 - x_0^2] + t \tag{4.1.13}$$

since $u(x,t)$ *satisfies the original equation* $u_t = u_{xx}$ *and the initial condition* $u(x,0) = p(x) = \frac{1}{2}[C(0) + 2x^2 - x_0^2] = x^2$ *so we have from these relations* $C(t) = 2t + c$ *and* $c = x_0^2$ *respectively, thus* $C(t) = 2t + x_0^2$ *substitute it in Eq(4.1.13) we get* $u(x,t) = 2t + x^2$.

4.2 Inverse Problem of Coefficient Identification

In this section, the application of the ADM is discussed for solving an inverse problem of coefficient identification [18, 35].
Consider the following inverse parabolic problem

$$u_t = \Delta u(x,t) + p(t)u(x,t) + \phi(x,t), \quad 0 < t < T, \quad x \in \Omega \tag{4.2.1}$$

where $\Omega = [0,1]^d$ is the special domain of the problem and its boundary is $\partial\Omega$, $x = (x_1,\dots,x_d)$, $\phi(x,t)$ is the source function and $p(t)$ is a control parameter.

In this problem both of function $u(x,t)$ and $p(t)$ are unknown, assume the above problem is subject to the initial condition

$$u(x,0) = f(x), \qquad x \in \Omega$$

and with boundary conditions

$$u(x,t) = h(x,t), \qquad 0 < t < T, \quad x \in \partial\Omega.$$

To solve this problem we must have extra condition in a point inside Ω, so let us define an additional condition at $x_0 \in \Omega$ as:

$$u(x_0,t) = E(t), \quad T > t > 0$$

The Solution Technique:

First, we will use the following transform in order to get a partial differential equation with only one unknown function.

$$\omega(x,t) = u(x,t)r(t), \tag{4.2.2}$$

where

$$r(t) = \exp(-\int_0^t p(s)ds). \tag{4.2.3}$$

From the (4.2.2) we find the values of u_t and Δu as follows

$$u_t = \frac{r(t)\omega_t - r'(t)\omega(x,t)}{r^2(t)} \tag{4.2.4}$$

and

$$\Delta u(x,t) = \frac{\Delta\omega(x,t)}{r(t)} \tag{4.2.5}$$

also from (4.2.3) we find that $p(t)$ is given by

$$-\int_0^t p(s)ds = lnr(t)$$

$$p(t) = -\frac{r'(t)}{r(t)} \tag{4.2.6}$$

$$\tag{4.2.7}$$

substitute (4.2.4), (4.2.5) and (4.2.6) in (4.2.1) to get

$$\omega_t(x,t) = \Delta\omega(x,t) + r(t)\phi(x,t) \tag{4.2.8}$$

with the initial condition

$$\omega(x,0) = u(x,0) = f(x), \qquad x \in \Omega$$

and boundary conditions

$$\omega(x,t) = u(x,t)r(t) = h(x,t)r(t), \quad 0 < t < T, \ \ x \in \partial\Omega$$

and the additional condition is

$$\omega(x_0,t) = u(x,t)r(t) = E(t)r(t), \quad 0 < t < T, \ \ x_0 \in \Omega$$

from above the relation of finding $r(t)$ is given by

$$r(t) = \frac{\omega(x_0,t)}{E(t)}. \tag{4.2.9}$$

From these information we conclude that the new partial differential equation is direct problem and we can solve it using ADM.
Apply $L_t^{-1} = \int_0^t (.)d\tau$ to both sides of (4.2.8) to get that:

$$\omega(x,t) = \omega(x,0) + \int_0^t (\Delta\omega(x,\tau) + r(\tau)\phi(x,\tau))d\tau$$

since $\omega(x,t) = \sum_{n=0}^{\infty} \omega_n(x,t)$ then

$$\sum_{n=0}^{\infty} \omega_n(x,t) = \omega(x,0) + \int_0^t (\Delta \sum_{n=0}^{\infty} \omega_n(x,t) + r_n(\tau)\phi(x,\tau))d\tau$$

since from (4.2.9) we have

$$r_n(t) = \frac{\omega_n(x_0,t)}{E(t)}. \tag{4.2.10}$$

Then the recursive relationship is

$$\omega_0(x,t) = \omega(x,0)$$

$$\omega_{n+1}(x,t) = \int_0^t (\Delta\omega_n(x,\tau) + r_n(\tau)\phi(x,\tau))d\tau, \quad n = 0,\ 1,\ 2, \dots \qquad (4.2.11)$$

after finding $\omega(x,t)$ and $r(t)$ from (4.2.11) and (4.2.10), respectively, we used them to find $u(x,t)$ and $p(t)$.

Example 4.2.1. *[35] Consider this inverse parabolic problem*

$$u_t = u_{xx}(x,t) + p(t)u(x,t) + e^{-t^2}(\pi^2 - (1+t)^2)(cos(\pi x) + sin(\pi x)), \quad 0 < t < T, \quad 0 < x < 1$$

subject to the initial condition

$$u(x,0) = cos(\pi x) + sin(\pi x), \qquad 0 < x < 1$$

and with boundary conditions

$$u(0,t) = e^{-t^2}, \qquad u(1,t) = -e^{-t^2}, \qquad 0 < t < T.$$

The extra additional condition is

$$u(x_0,t) = e^{-t^2}(cos(\pi x) + sin(\pi x))$$

Solution:

Using the above technique we get that

$$\begin{aligned}
\omega_0(x,t) &= cos(\pi x) + sin(\pi x) \\
r_0(t) &= \frac{\omega_0(x_0,t)}{E(t)} = \frac{cos(\pi x) + sin(\pi x)}{e^{-t^2}(cos(\pi x) + sin(\pi x))} = e^{t^2} \\
\omega_1(x,t) &= \int_0^t ((\omega_0)_{xx}(x,t) + r_0(\tau)\phi(x,\tau))d\tau \\
&= \int_0^t [(-\pi^2)(cos(\pi x) + sin(\pi x)) + e^{\tau^2}e^{-\tau^2}(\pi^2 - (1+t)^2)(cos(\pi x) + sin(\pi x))]d\tau \\
&= \int_0^t (-(1+\tau)^2)(cos(\pi x) + sin(\pi x))d\tau = (\frac{-(1+t)^3}{3} + \frac{1}{3})(cos(\pi x) + sin(\pi x)) \\
r_1(t) &= \frac{\omega_1(x_0,t)}{E(t)} = e^{t^2}(\frac{-(1+t)^3}{3}) + \frac{1}{3})
\end{aligned}$$

after many terms we have $r(t) = e^{t^2}$ *and* $\omega(x,t) = cos(\pi x) + sin(\pi x)$*, thus from* (4.2.2) *and* (4.2.3) *we get* $u(x,t) = e^{t^2}(cos(\pi x) + sin(\pi x))$ *and* $p(t) = 1 + t^2$.

4.3 The Inverse Conductivity problem

In [25, 10, 30, 26] researchers worked to give numerical solution of the inverse conductivity problem. Let $\Omega \in R^2$ be an open and bounded subset with smooth boundary $\partial\Omega$. Assuming that there are no sources or sinks of current in Ω, the application of a voltage potential f on $\partial\Omega$ induces a voltage potential u inside Ω defined as the unique solution of the boundary value problem

$$\nabla.(\gamma\nabla u) = 0 \ \ in \ \ \Omega, \ \ \ u = f \ \ on \ \ \partial\Omega$$

where $\gamma : \Omega \to (0,\infty)$ is measurable and bounded away from zero to infinity. Often we can assume that γ is constant near $\partial\Omega$ always we take $\gamma = 1$ near $\partial\Omega$, then if $f \in C^2(\partial\Omega)$, the solution $u \in C^1$ near $\partial\Omega$, so the following classical Neumann data are well defined

$$\gamma\frac{\partial u}{\partial \nu} \mid_{\partial\Omega}= h,$$

where h is a function in $C^2(\partial\Omega)$. To solve this problem Kim Knudsen [30] reduced the problem to Schrödinger equation.

$$\begin{aligned}(\Delta - q)\upsilon = 0 \ \ in \ \ \Omega,\\ \upsilon \mid_{\partial\Omega}= \gamma^{1/2}u \mid_{\partial\Omega}= u \mid_{\partial\Omega}= f\end{aligned} \tag{4.3.1}$$

where q, υ are given by

$$\upsilon = \gamma^{\frac{1}{2}}u$$

$$q = \gamma^{-1/2}\Delta\gamma^{1/2}$$

since $\gamma = 1$ near $\partial\Omega$ and $u \mid_{\partial\Omega} = f$, In this section we will apply the Adomian decomposition method to the problem

$$(\Delta - q)\upsilon = 0 \quad in \quad \Omega,$$
$$\upsilon \mid_{\partial\Omega} = f, \quad \partial_\nu \upsilon = h \quad on \quad \partial\Omega$$

from $\partial_\nu \upsilon = h$ we can get approximation of $\partial\upsilon$, which is assumed to be ζ. In $\mathbb{R}^{\not\vDash}$, $\upsilon = \upsilon(x, y)$ so equation(4.3.1) is

$$\partial_{xx}\upsilon + \partial_{yy}\upsilon - q\upsilon(x, y) = 0 \tag{4.3.2}$$

since $\gamma : \Omega \to (0, \infty)$, $\partial_\nu \upsilon = h(y)$ and $\upsilon \mid_{\partial\Omega} = f$. Rewrite equation(4.3.2) in the operator form

$$L_{xx}\upsilon + L_{yy}\upsilon - R\upsilon = 0 \tag{4.3.3}$$

where $L_{xx} = \frac{\partial^2}{\partial x^2}$, $L_{yy} = \frac{\partial^2}{\partial y^2}$ and $R\upsilon = q\upsilon$. Applying $L_{yy}^{-1} = \int_0^y \int_0^y$ to both sides of equation(4.3.3)

$$L_{yy}^{-1} L_{yy}\upsilon = L_{yy}^{-1} q(x)\upsilon - L_{yy}^{-1} L_{xx}\upsilon$$
$$\int_0^y \int_0^y (\frac{\partial^2 \upsilon}{\partial s^2}) ds ds = \int_0^y (\frac{\partial \upsilon}{\partial s} - \upsilon_y(x, 0)s) ds$$

then we have

$$\upsilon(x, y) - \upsilon_y(x, 0)x - \upsilon(x, 0) = L_{yy}^{-1} q(x)\upsilon - L_{yy}^{-1} L_{xx}\upsilon \tag{4.3.4}$$

using the boundary condition $\upsilon(x, 0) = f$ and $\upsilon_y(x, 0) = \zeta$ at $y = 0$, substituting in (4.3.4)

$$\upsilon(x, y) = \zeta y + f + L_{yy}^{-1} q(x)\upsilon - L_{yy}^{-1} L_{xx}\upsilon. \tag{4.3.5}$$

As we knew Adomian decomposition method gives the solution $\upsilon(x, y)$ as infinite series,

$$\upsilon(x, y) = \sum_{n=0}^{\infty} \upsilon_n(x, y)$$

substituting in (4.3.5) to get

$$\sum_{n=0}^{\infty} \upsilon_n(x,y) = \zeta y + f + L_{yy}^{-1} q(x) \sum_{n=0}^{\infty} \upsilon_n(x,y) - L_{yy}^{-1} L_{xx} \sum_{n=0}^{\infty} \upsilon_n(x,y). \qquad (4.3.6)$$

from equation(4.3.6) define $\upsilon_0 = \zeta y + f = f(x) + \zeta(x)y$ and

$$\upsilon_{n+1}(x,y) = L_{yy}^{-1} q(x)\upsilon_n - L_{yy}^{-1} L_{xx}\upsilon_n \quad for \quad n = 0,\ 1,\ 2, \ldots.$$

The first three terms are

$$\begin{aligned}
\upsilon_1 &= q(x)L_{yy}^{-1}\upsilon_0 - L_{yy}^{-1}L_{xx}\upsilon_0 \\
&= q(x)L_{yy}^{-1}(f(x) + \zeta(x)y) - L_{yy}^{-1}L_{xx}(f(x) + \zeta(x)y) \\
&= q(x)(f(x)\frac{y^2}{2} + \zeta\frac{y^3}{3!}) - L_{yy}^{-1}(f'' + \zeta'' y) \\
&= q(x)(f(x)\frac{y^2}{2} + \zeta\frac{y^3}{3!}) - (f''\frac{y^2}{2} + \zeta''\frac{y^3}{3!}) \\
\upsilon_2 &= q^2(x)(f(x)\frac{y^4}{4!} + \zeta\frac{y^5}{5!}) - (f^{(4)}\frac{y^4}{4!} + \zeta^{(4)}\frac{y^5}{5!}) \\
\upsilon_3 &= q^3(x)(f(x)\frac{y^6}{6!} + \zeta\frac{y^7}{7!}) - (f^{(6)}\frac{y^6}{6!} + \zeta^{(6)}\frac{y^7}{7!})
\end{aligned}$$

then,

$$\begin{aligned}
\upsilon(x,y) &= \upsilon_0 + \upsilon_1 + \upsilon_2 + \upsilon_3 + \cdots \\
&= f(x) + \zeta(x)y + q(x)(f(x)\frac{y^2}{2} + \zeta\frac{y^3}{3!}) - (f''\frac{y^2}{2} \\
&+ \zeta''\frac{y^3}{3!}) + q^2(x)(f(x)\frac{y^4}{4!} + \zeta\frac{y^5}{5!}) \\
&- (f^{(4)}\frac{y^4}{4!} + \zeta^{(4)}\frac{y^5}{5!}) + q^3(x)(f(x)\frac{y^6}{6!} \\
&+ \zeta\frac{y^7}{7!}) - (f^{(6)}\frac{y^6}{6!} + \zeta^{(6)}\frac{y^7}{7!}) + \cdots
\end{aligned}$$

by reorder the terms of $\upsilon(x,y)$ we get that

$$\begin{aligned}
\upsilon(x,y) = {} & f(1 + q(x)\frac{y^2}{2!} + q^2(x)\frac{y^4}{4!} + q^3\frac{y^6}{6!} + \cdots) \\
& + \zeta(y + q(x)\frac{y^3}{3!} + q^2(x)\frac{y^5}{5!} + q^3\frac{y^7}{7!} + \cdots) - (f''\frac{y^2}{2!} + f^{(4)}\frac{y^4}{4!} + f^{(6)}\frac{y^6}{6!} + \cdots) \\
& - (\zeta''\frac{y^3}{3!} + \zeta^{(4)}\frac{y^5}{5!} + \zeta^{(6)}\frac{y^7}{7!} + \cdots)
\end{aligned}$$

this solution can be written as summation of these series

$$\upsilon(x,y) = f\sum_{0}^{\infty} q^n \frac{y^{2n}}{2n!} + \zeta \sum_{0}^{\infty} q^n \frac{y^{2n+1}}{2n+1!} - \sum_{1}^{\infty} f^{(2n)} \frac{y^{2n}}{2n!} - \sum_{1}^{\infty} \zeta^{(2n)} \frac{y^{2n+1}}{2n+1!}$$

$\upsilon(x,y)$ is approximately

$$\upsilon(x,y) \propto \exp(iq.y) + w(x,y) \tag{4.3.7}$$

Since $\upsilon(x,y)$ is bounded and is the unique solution of the Schrödinger equation so it has this formula

$$\upsilon(x,y) = \exp(ik.x) \quad as \mid x \mid \to \infty \tag{4.3.8}$$

where k is a complex orthonormal vector, $k \neq 0$ and $x \in \mathbb{R}$. So from (4.3.8) and (4.3.7)

$$exp(ik.x) \propto exp(iq.y) + w(x,y)$$

but as $\mid x \mid \to \infty$ must $\parallel w(x,y) \parallel \to 0$ in order to keep $\upsilon(x,y)$ bounded. Thus, we have

$$exp(ik.x) \propto exp(iq.y)$$

$\Rightarrow q \propto k.x/y$, from $q = \gamma^{-1/2}\Delta\gamma^{1/2}$ and $u = \gamma^{-1/2}\upsilon$ we can find γ and u of the conductivity problem respectively.

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